Math 2153, Exam #2, Spring 2014 Name KEY

Instructions: Show all work. Use exact answers unless specifically asked to round. You may check your answers in the calculator, but you must show work to get full credit. Incorrect answers with no work will receive no credit. Be sure to complete all the requested elements of each problem.

1. Find both partial derivatives for $f(x,y) = x^y$. (8 points)

$$\frac{\partial f}{\partial x} = y x^{y-1}$$

$$\frac{\partial f}{\partial y} = x^y \cdot \ln x$$

2. Find f_{xxy} and f_{xxx} for the function $f(x,y) = (y + ix)^3$. (12 points)

$$\frac{\partial f}{\partial x} = 3(y+ix)^2 \cdot i = 3i(y+ix)^2$$

$$\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] = 6i(y+ix)\cdot i = -6(y+ix) = -6y - 6ix$$

$$f_{xxx} = -6i$$

$$f_{xxy} = -6$$

3. Find the total differential for the function $w(x,y,z,t) = \frac{yt}{z-x}$ at the point (0,1,2,3) and use it to approximate the value of the function at (0.1,0.95,2.01,2.98). (8 points)

$\Delta x = .1$ $\Delta y = -.05$, $\Delta z = .01$ $\Delta t = -.02$

$$\frac{\partial w}{\partial x} = \frac{yt}{(z-x)^2} \qquad \frac{\partial w}{\partial y} = \frac{t}{z-x}$$

$$\frac{\partial w}{\partial z} = \frac{-yt}{(z-x)^2} \qquad \frac{\partial w}{\partial t} = \frac{y}{z-x}$$

$$dw = \frac{3}{4}(.1) + \frac{3}{2}(-.05) + \left(\frac{-3}{4}\right)(.01) + \frac{1}{2}(-.02) = -.0175$$

$$\frac{\partial w}{\partial x}\cdot\Delta x + \frac{\partial w}{\partial y}\cdot\Delta y + \frac{\partial w}{\partial z}\Delta z + \frac{\partial w}{\partial t}\Delta t$$

$$w(0,1,2,3) = \frac{3}{2}$$

$$w(.1,.95,2.01,2.98) \approx 1.5 - .0175 = \boxed{1.4825}$$

4. Find a double integral to compute the area of the region bounded by $y = 4 - x^2$ and $y = x^2 - 4$. Sketch the region. Integrate your integral to find the area. (10 points)

$$\int_{-2}^{2}\int_{x^2-4}^{4-x^2} dy\, dx = \int_{-2}^{2} 4 - x^2 - x^2 + 4\, dx$$

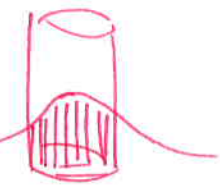
$$= \int_{-2}^{2} 8 - 2x^2\, dx = 8x - \frac{2}{3}x^3 \Big|_{-2}^{2} = 16 - \frac{2}{3}\cdot 8$$

$$-(-16) + (-\frac{2}{8}\cdot 8) =$$

$$32 - \frac{32}{3} = \frac{96-32}{3} = \boxed{\frac{64}{3}}$$

5. Find the volume of the solid inside the cylinder $x^2 + y^2 = 9$ above the xy-plane and below the function $z = \frac{1}{1+(x^2+y^2)^2}$. Set up a double or triple integral and compute the result. (13 points)

$$z = \frac{1}{1+r^4} \qquad r = 3$$



$$\int_0^{2\pi}\int_0^{3} \frac{1}{1+r^4}\, r\, dr\, d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^{9} \frac{du}{1+u^2}\, d\theta$$

$u = r^2$

$\frac{1}{2} du = \cancel{2} r\, dr$

$$= \frac{1}{2}\int_0^{2\pi} \arctan u \Big|_0^9 d\theta = \frac{1}{2}\int_0^{2\pi} \arctan 9\, d\theta =$$

$$\frac{1}{2}\cdot 2\pi \arctan 9 = \pi \arctan 9 \approx 4.587$$

6. For $\vec{F}(x,y,z) = xz\hat{\imath} - \frac{y^2}{z}\hat{\jmath} + x^2y\hat{k}$, $\vec{G}(x,y,z) = x^3\hat{\imath} + \ln xy\,\hat{\jmath} - \frac{1}{z}\hat{k}$ and $f(x,y,z) = x^2y + e^y + yz - 11$, find each of the following. (8 points each)

a. ∇f

$$\nabla f = \langle 2xy,\ x^2 + e^y + z,\ y \rangle$$

b. $\nabla \times \vec{G}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^3 & \ln x + \ln y & -z^{-1} \end{vmatrix} = (0-0)\hat{i} - (0-0)\hat{j} + \left(\frac{1}{x} - 0\right)\hat{k}$$

$$= \frac{1}{x}\hat{k}$$

c. $\nabla^2 f$

$$\nabla \cdot \nabla f = 2y + e^y$$

d. $\nabla \cdot \vec{F}$

$$z - \frac{2y}{z}$$

7. Sketch the gradient of the function $f(x, y) = x^2 + y^2 + 2x - 6y + 15$. Clearly indicate where each component of the gradient is equal to zero. Label the general direction of the gradient in each region, and use that information to sketch at least three level curves. (10 points)

$\langle 2x+2, 2y-6 \rangle$

$2x+2=0$ $\quad$ $2y-6=0$

$2x=-2$ $\quad$ $2y=6$

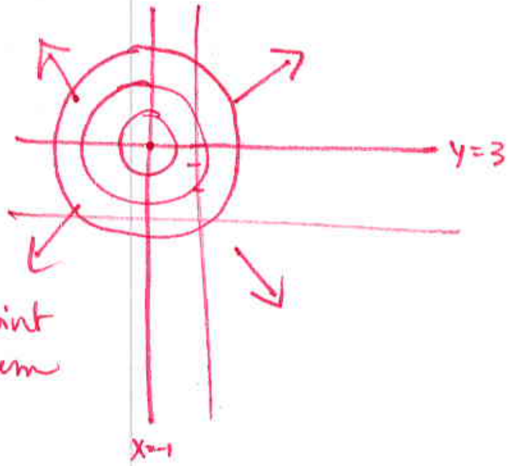
$x=-1$ $\quad$ $y=3$

$(4,5)$ $\langle 10, 4 \rangle$

$(-2,5)$ $\langle -2, 4 \rangle$

$(-2,0)$ $\langle -2, -6 \rangle$

$(4,0)$ $\langle 10, -6 \rangle$



Critical point is a minimum

8. Integrate. (8 points each)

a. $\int_1^3 \int_x^4 \int_{x+y}^{16-x^2} xy\,dz\,dy\,dx$

$$xyz\Big|_{x+y}^{16-x^2} = xy(16-x^2-x-y) = 16xy - x^3y - x^2y - xy^2$$

$$\int_1^3\int_x^4 16xy - x^3y - x^2y - xy^2\,dy\,dx = 8xy^2 - \frac{x^3y^2}{2} - \frac{x^2y^2}{2} - \frac{xy^3}{3}\Big|_x^4 =$$

$$\int_1^3 128x - 8x^3 - 8x^2 - \frac{64x}{3} - 8x^3 + \frac{1}{2}x^5 + \frac{1}{2}x^4 + \frac{1}{3}x^4\,dx = \int_1^3 \frac{1}{2}x^5 + \frac{5}{6}x^4 - 16x^3 - 8x^2 + \frac{320}{3}x\,dx$$

$$= \boxed{\frac{415}{3}}$$

b. $\int_0^\pi \int_0^\pi \int_0^{\sin\varphi} \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$

$$\frac{1}{3}\rho^3\Big|_0^{\sin\varphi} = \frac{1}{3}\sin^3\varphi$$

$$\frac{1}{3}\int_0^\pi\int_0^\pi \sin^4\varphi\,d\varphi\,d\theta = \frac{1}{12}\int_0^\pi\int_0^\pi (1-\cos 2\varphi)^2\,d\varphi\,d\theta = \frac{1}{12}\int_0^\pi\int_0^\pi 1 - 2\cos 2\varphi + \cos^2 2\varphi\,d\varphi\,d\theta$$

$$\frac{1}{2}(1+\cos 4\varphi)$$

$$= \frac{1}{12}\int_0^\pi\int_0^\pi \frac{3}{2} - 2\cos 2\varphi + \frac{1}{2}\cos 4\varphi\,d\varphi\,d\theta = \frac{1}{12}\int_0^\pi \frac{3}{2}\varphi - \sin 2\varphi + \frac{1}{8}\sin 4\varphi\Big|_0^\pi d\theta$$

$$\frac{1}{12}\int_0^\pi \frac{3}{2}\pi\,d\theta = \frac{1}{12}\cdot\frac{3}{2}\pi\cdot\pi = \boxed{\frac{1}{8}\pi^2}$$

c. $\int_0^{2\pi}\int_0^3\int_0^{2-r} r^2\,dz\,dr\,d\theta$

$$r^2z\Big|_0^{2-r} = 2r^2 - r^3$$

$$\int_0^{2\pi}\int_0^3 2r^2 - r^3\,dr\,d\theta = \frac{2}{3}r^3 - \frac{1}{4}r^4\Big|_0^3 = 18 - \frac{81}{4} = -\frac{9}{4}$$

$$\int_0^{2\pi} -\frac{9}{4}\,d\theta = -\frac{9}{\cancel{4}_2}\cdot\cancel{2}\pi = \boxed{\frac{-9\pi}{2}}$$

9. Find the potential function, if it exists, of the vector field $\vec{F}(x,y,z) = 2xe^{x^2}\hat{\imath} + (z-3y^2)\hat{\jmath} + \left(y+\frac{1}{z}\right)\hat{k}$. If it does not exist, verify that it doesn't exist by showing that the curl is non-zero. (10 points)

$$\int 2xe^{x^2}dx = e^{x^2} + G(y,z)$$

$$\int z - 3y^2\,dy = zy - y^3 + H(x,z)$$

$$\int y + \frac{1}{z}\,dz = yz + \ln|z| + I(x,y)$$

$$\boxed{e^{x^2} + yz - y^3 + \ln|z| + K}$$

10. Determine if the vector field $\vec{G}(x,y) = 5xy\hat{\imath} + \left(\frac{5}{2}x^2 + 1\right)\hat{\jmath} + xz\hat{k}$ is conservative. (6 points)

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 5xy & \frac{5}{2}x^2+1 & xz \end{vmatrix} = (0-0)\hat{\imath} - (z-0)\hat{\jmath} + (5x-5x)\hat{k}$$

$$0\hat{\imath} - z\hat{\jmath} + 0\hat{k}$$

no conservative since $\vec{\nabla} \times \vec{G} \neq 0$.

11. Show that the mixed partial derivatives for $g(x,y) = e^{xy} + x\ln x - y^2 \sin x$ are equal. (10 points)

$$g_x = ye^{xy} + \cancel{x} \cdot \frac{1}{\cancel{x}} + \ln x - y^2 \cos x$$

$$g_{xy} = e^{xy} + xye^{xy} - 2y\cos x$$

$$g_y = xe^{xy} - 2y\sin x$$

$$g_{yx} = e^{xy} + xye^{xy} - 2y\cos x$$

$= \checkmark$