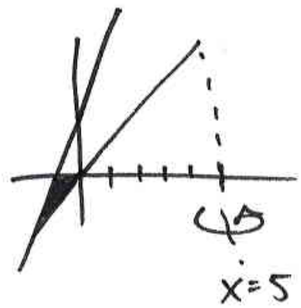
Math 1152, Final Exam, Spring 2014 Name KEY

Instructions: Show all work. Use exact answers unless specifically asked to round. You may check your answers in the calculator, but you must show work to get full credit. Incorrect answers with no work will receive no credit. Be sure to complete all the requested elements of each problem.

1. Find the volume of the solid form from revolving the region formed by the equations $y = x, y = 2x+1, y = 0$ around the line x=5. Use the method of your choice. Sketch the region. (15 points)

Shell method



$$V = 2\pi \int_{-1}^{0} (5-x)(2x+1-x)\,dx$$

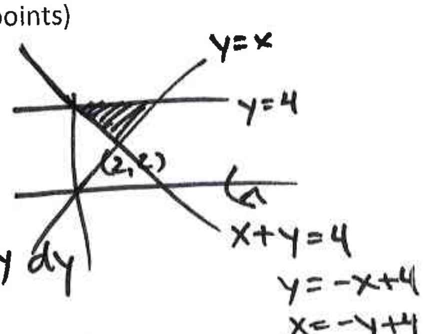
$$= 2\pi \int_{-1}^{0} (5-x)(x+1)\,dx =$$

$$= 2\pi \int_{-1}^{0} 5x + 5 - x^2 - x\,dx = 2\pi \int_{-1}^{0} -x^2 + 4x + 5\,dx =$$

$$2\pi \left[-\frac{x^3}{3} + 2x^2 + 5x \right]_{-1}^{0} = -2\pi \left[-\frac{(-1)^3}{3} + 2(-1)^2 + 5(-1) \right] =$$

$$-2\pi \left[\frac{1}{3} + 2 - 5 \right] = -2\pi \left[-8/3 \right] = \boxed{\frac{16\pi}{3}}$$

2. Find the volume of the solid formed from revolving the region given by $y = 4, x - y = 0, x + y = 4$ around the x-axis. Use the shell method. Sketch the region. (15 points)



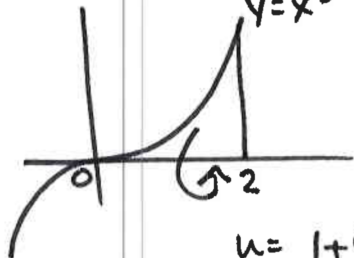
$$V = 2\pi \int_{2}^{4} y\,(y - (-y+4))\,dy =$$

$$= 2\pi \int_{2}^{4} y(2y-4)\,dy = 2\pi \int_{2}^{4} 2y^2 - 4y\,dy$$

$$= 2\pi \left[\frac{2}{3}y^3 - 2y^2 \right]_{2}^{4} = 2\pi \left[\frac{2}{3}\cdot 64 - 2\cdot 16 - \frac{2}{3}\cdot 8 + 2\cdot 4 \right]$$

$$= 2\pi \left[40/3 \right] = \boxed{\frac{80\pi}{3}}$$

3. Find the surface area of the shape formed by revolving the graph $y = x^3$ around the y-axis, on the interval [0,2]. Sketch the region. (15 points)



$f'(x) = 3x^2$

$[f'(x)] = 9x^4$

$$S = 2\pi \int_0^2 x^3 \sqrt{1+9x^4}\, dx$$

$u = 1+9x^4$

$du = 36x^3\, dx$

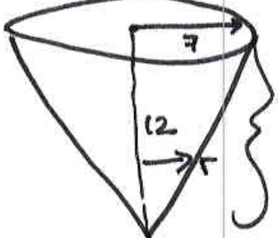
$x = 0 \Rightarrow u = 1$

$x = 2 \Rightarrow u = 1+9(16) = 145$

$$= \frac{2\pi}{36}\int_1^{145} u^{1/2}\, du = \frac{\pi}{18}\left[\frac{2}{3}u^{3/2}\right]_1^{145}$$

$$= \frac{\pi}{27}\left[145^{3/2} - 1\right]$$

4. Consider a conical tank with diameter 14 feet and a height of 12 feet. Find the amount of work needed to drain the tank if it is only half full. Assume that the fluid in the tank is water and it has a weight-density of 62.4 lbs./ft³. (20 points)



$\frac{12}{7} = \frac{h}{b}$

$r(y) = \frac{h}{b} y$

$\frac{12}{7} = \frac{y}{r(y)}$

$r(y) = \frac{7}{12} y$

$r(x)$ for slice $= \frac{12}{7} y$

Similar triangles

distance of slice traveled

$12 - y$

where y is height of slice

cylindrical slice

$\pi r^2 h = \pi \left[\frac{7}{12} y\right]^2 dy$

$= \frac{49\pi}{144} y^2\, dy$

weight = force =

$62.4 \cdot \frac{49}{144} \pi y^2$

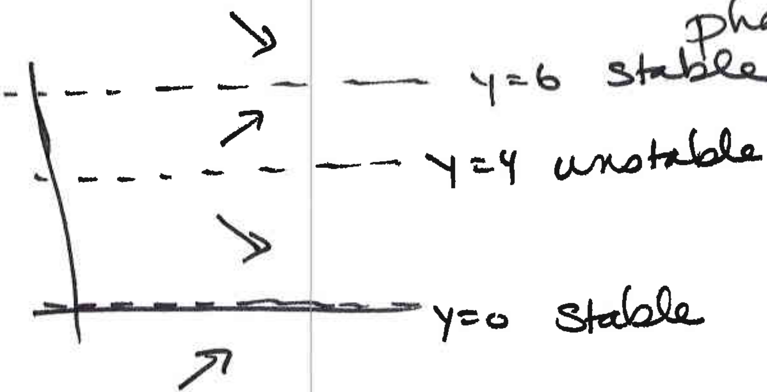
$$W = \int_0^{6 \leftarrow \text{half full}} 62.4 \cdot \frac{49}{144} \pi\, y^2 (12-y)\, dy = 62.4 \cdot \frac{49\pi}{144} \int_0^6 12y^2 - y^3\, dy =$$

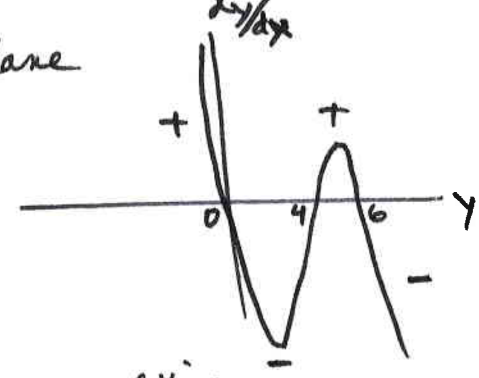
$$62.4 \cdot \frac{49\pi}{144}\left[4y^3 - \frac{1}{4}y^4\right]_0^6 = 62.4 \cdot \frac{49\pi}{144} * 540 = 11{,}466\pi \approx 36{,}021.5$$

foot-pounds

5. Given the differential equation $\frac{dy}{dx} = y(6-y)(y-4)$. Sketch the phase plane for the equation and use that information to graph the key features of the direction field such as the equilibria (steady state solutions) and the sign of the slope in each region. Label each equilibrium as stable, unstable or semi-stable. (15 points)

Phase plane





equilibria are
$y = 0, 4, 6$

6. Solve the separable differential equation $y' = \frac{x\sqrt{1-y^2}}{x^2+4}$. (12 points)

$$\int \frac{dy}{\sqrt{1-y^2}} = \int \frac{x\,dx}{x^2+4}$$

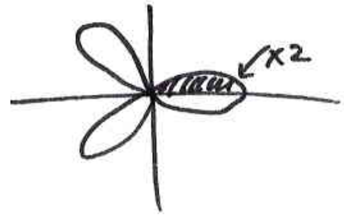
$$\arcsin y = \frac{1}{2}\ln|x^2+4| + C$$

$$\arcsin y = \ln\sqrt{x^2+4} + C \Rightarrow A\ln\sqrt{x^2+4}$$

$$y = \sin\left[A\ln\sqrt{x^2+4}\right]$$

7. Find the area of one petal of the graph $r = 4\cos 3\theta$. (10 points)

$$4\cos 3\theta = 0$$
$$\cos 3\theta = 0$$
$$3\theta = \pi/2$$
$$\theta = \pi/6$$



$$A = \frac{2}{2}\int_0^{\pi/6} (4\cos 3\theta)^2 \, d\theta = 16\int_0^{\pi/6} \cos^2 3\theta \, d\theta = 8\int_0^{\pi/6} 1 + \cos 6\theta \, d\theta$$

$$= 8\left[\theta + \frac{1}{6}\sin 6\theta \Big|_0^{\pi/6}\right] = 8\left[\pi/6 + \frac{1}{6}\cdot 0 - 0 - 0\right] = \boxed{\frac{4\pi}{3}}$$

8. Find the angle between the vectors $\langle 7,3,1\rangle$ and $\langle 9,-5,-2\rangle$. State your answers in radians to 4 decimal places. (8 points)

$$\vec{u}\cdot\vec{v} = 7\times 9 + 3(-5) + 1(-2) = 46$$

$$\|\vec{u}\| = \sqrt{49+9+1} = \sqrt{59}$$

$$\|\vec{v}\| = \sqrt{81+25+4} = \sqrt{110}$$

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|} = \frac{46}{\sqrt{59}\cdot\sqrt{110}} = .570999$$

$$\cos^{-1}(.570999) \approx .9631 \text{ radians}$$

9. Find the cross product of the vectors $2\hat{\imath} + 3\hat{\jmath} - 2\hat{k}$ and $\hat{\imath} + 2\hat{\jmath} - \hat{k}$. Find the magnitude of the vector. (10 points)

$$\vec{u} \times \vec{v} \quad \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & -2 \\ 1 & 2 & -1 \end{vmatrix} = (-3+4)\hat{\imath} - (-2+2)\hat{\jmath} + (4-3)\hat{k}$$

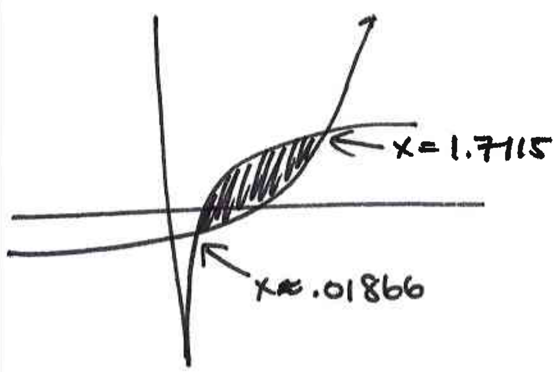
$$= \hat{\imath} + \hat{k}$$

$$\| \vec{u} \times \vec{v} \| = \sqrt{2}$$

10. Set up, but do not integrate, the integral to find the area of the region bounded by $y = e^x - 2$ and y= $\ln x + 3$. Find your limits numerically and state them to 4 decimal places. (8 points)

$$A \approx \int_{.01866}^{1.71152} (\ln x + 3) - (e^x - 2)\, dx$$

$$= \int_{.01866}^{1.71152} \ln x + 5 - e^x \, dx$$



11. Integrate. Use any technique which is appropriate. (12 points each)

a. $\int \cosh^3 x \sinh x \, dx$

$u = \cosh x$

$du = \sinh x$

$$\int u^3 du = \frac{1}{4} \cosh^4 x + C$$

b. $\int x \tan^2 x \, dx$ $= \int x(1-\sec^2 x)\,dx = \int x - x\sec^2 x \, dx$

$\frac{1}{2}x^2 - \int x\sec^2 x \, dx$ $= \frac{1}{2}x^2 - x\tan x + \int \tan x \, dx$

$u = x \quad dv = \sec^2 x$

$du = dx \quad v = \tan x \, dx$

$= \frac{1}{2}x^2 - x\tan x - \ln|\cos x| + C$

c. $\int \frac{\sqrt{9+x^2}}{x^2} dx$

$x = 3\tan\theta \quad x^2 = 9\tan^2\theta$

$dx = 3\sec^2\theta \, d\theta$

$9 + x^2 = 9(1+\tan^2\theta) = 9\sec^2\theta$

$\sqrt{9+x^2} = 3\sec\theta$

$$\int \frac{3\sec\theta \cdot 3\sec^2\theta \, d\theta}{9\tan^2\theta} = \int \frac{\sec^3\theta}{\tan^2\theta} d\theta = \int \frac{1}{\cos^3\theta} \cdot \frac{\cos^2\theta}{\sin^2\theta} d\theta$$

$$= \int \frac{1}{\cos\theta} \cdot \frac{1}{\sin^2\theta} d\theta = \int \frac{1}{\cos\theta} \cdot \frac{1}{1-\cos^2\theta} d\theta = \int \frac{1}{\cos\theta} \cdot \frac{1}{1+\cos\theta} \cdot \frac{1}{1-\cos\theta} d\theta$$

$$\frac{A}{\cos\theta} + \frac{B}{1+\cos\theta} + \frac{C}{1-\cos\theta} = 1 \Rightarrow A(1-\cos^2\theta) + B\cos\theta(1-\cos\theta) + C\cos\theta(1+\cos\theta) = 1$$

Continued $\longrightarrow$

d. $\int_0^\infty xe^{-x^2} dx$ (if the integral does not converge, be sure to state that clearly)

$u = -x^2 \quad x = 0 \Rightarrow u = 0$

$du = -2x\,dx \quad x = \infty \Rightarrow u = -\infty$

$$-\frac{1}{2}\int_0^{-\infty} e^u \, du = \frac{1}{2}\int_{-\infty}^0 e^u \, du = \frac{1}{2}[e^0 - 0] = \boxed{\frac{1}{2}}$$

Converges

11c cont'd.

$\theta = \pi/2 \qquad A + 0 + 0 = 1 \qquad A = 1$

$\theta = 0 \qquad 0 + 0 + 2C = 1 \qquad C = 1/2$

$\theta = \pi \qquad 0 + -2B + 0 = 1 \qquad B = -1/2$

$$\int \frac{1}{\cos\theta} + \frac{-1/2}{1+\cos\theta} + \frac{1/2}{1-\cos\theta}\, d\theta =$$

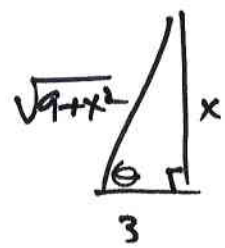
$$\int \sec\theta + \frac{-1/2 + 1/2\cos\theta + 1/2 + 1/2\cos\theta}{1-\cos^2\theta}\, d\theta =$$

$$\int \sec\theta + \frac{\cos\theta}{\sin^2\theta}\, d\theta = \int \sec\theta + \csc\theta\cot\theta \, d\theta$$

$$\frac{\cos\theta}{\sin\theta} \cdot \frac{1}{\sin\theta}$$

$$= \ln|\sec\theta + \tan\theta| - \csc\theta + C$$

$$\ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| - \frac{\sqrt{9+x^2}}{x} + C$$



$\tan\theta = \frac{x}{3}$

e. $\int x^5 \sin 2x \, dx$

$\pm$	u	dv
$+$	x^5	$\sin 2x$
$-$	$5x^4$	$-\frac{1}{2}\cos 2x$
$+$	$20x^3$	$-\frac{1}{4}\sin 2x$
$-$	$60x^2$	$\frac{1}{8}\cos 2x$
$+$	$120x$	$\frac{1}{16}\sin 2x$
$-$	120	$-\frac{1}{32}\cos 2x$
	0	$-\frac{1}{64}\sin 2x$

$$-\frac{1}{2}x^5\cos 2x + \frac{5}{4}x^4\sin 2x + \frac{5}{2}x^3\cos 2x - \frac{15}{4}x^2\sin 2x - \frac{15}{4}x\cos 2x$$
$$+ \frac{15}{8}\sin 2x + C$$

12. Find the limit of the sequence $a_n = \frac{1}{2}\arctan(n)$ if it exists. If it does not, state that it diverges. (8 points)

$$\lim_{n\to\infty} \frac{1}{2}\arctan(n) = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}$$

13. Find the value that the series converges to (all of these converge to some known value). (10 points each)

a. $\sum_{k=1}^{\infty} \frac{2}{(2k+3)(2k+1)}$

$$\frac{A}{2k+3} + \frac{B}{2k+1} = \frac{2}{(2k+3)(2k+1)}$$

$$A(2k+1) + B(2k+3) = 2 \qquad k = -\tfrac{1}{2}$$

$$2B = 2$$
$$B = 1 \qquad k = -\tfrac{3}{2}$$

$$-2A = 2 \Rightarrow A = -1$$

$$\sum_{k=1}^{\infty} \frac{1}{2k+1} - \frac{1}{2k+3} =$$

$$\lim_{k\to\infty} \frac{1}{3} - \frac{1}{2k+3} = \frac{1}{3}$$

b. $\sum_{k=0}^{\infty} 3^n 5^{-n}$ $= \sum_{k=0}^{\infty} \left(\frac{3}{5}\right)^n \qquad r = \frac{3}{5}$

$$S = \frac{1}{1 - 3/5} = \frac{5}{2}$$

c. $\sum_{n=1}^{\infty} \frac{3^n}{n!}$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n!} = e^x - \frac{x^0}{0!} = e^x - 1$$

$\therefore$ if $x = 3$

$$\sum_{n=1}^{\infty} \frac{3^n}{n!} = \boxed{e^3 - 1}$$

14. Determine if the infinite series converge or diverge. Explain your reasoning, and which test you used to determine it. (11 points each)

a. $\sum_{k=2}^{\infty} \frac{ln^2(k)}{k}$

$$\int_2^{\infty} \frac{\ln^2 k}{k}\, dk =$$

$u = \ln k$
$du = \frac{1}{k} dk$

$$\int_{\ln 2}^{\infty} u^2\, du = \infty$$

diverges by the integral test

b. $\sum_{k=1}^{\infty} ke^{-k^2}$

$$\int_1^{\infty} ke^{-k^2} dk$$

$u = -k^2 \Rightarrow u = -1, u = -\infty$
$du = -2k\,dk$

$$-\frac{1}{2}\int_{-1}^{-\infty} e^u\, du = \frac{1}{2}\int_{-\infty}^{-1} e^u\, du =$$

$$\frac{1}{2}[e^{-1} - 0] = \frac{1}{2e}$$

Converges by the integral test

(ratio test also works here, as do the root test)

c. $\sum_{k=1}^{\infty} \frac{k!}{k^k}$ $\lim_{k\to\infty} \left| \frac{(k+1)!}{(k+1)^{k+1}} \cdot \frac{k^k}{k!} \right| = \lim_{k\to\infty} \left| \frac{\cancel{(k+1)} \cdot k^k}{(k+1)^k \cancel{(k+1)}} \right| =$

$$\lim_{k\to\infty} \left(\frac{k}{k+1} \right)^k = \lim_{k\to\infty} \left(1 - \frac{1}{k}\right)^k = \frac{1}{e} < 1$$

Converges by the ratio test

d. $\sum_{k=1}^{\infty} (-1)^k k^{1/k}$

$\lim_{k\to\infty} k^{1/k} = 1$ diverges by the nth term test (or the alternating series test)

15. Rewrite the expression $y = \frac{3x}{4-5x^2}$. (12 points)

$$\frac{a}{1-r} = a \sum_{k=0}^{\infty} r^n$$

$r = 5x^2/4$

$a = 3x/4$

$$y = \frac{3x/4}{4/4 - 5x^2/4} = \frac{\frac{3}{4}x}{1 - \frac{5}{4}x^2}$$

$$\frac{3}{4}x \sum_{k=0}^{\infty} \left(\frac{5}{4}x^2\right)^n = \sum_{k=0}^{\infty} \frac{3}{4} \cdot \frac{5^n}{4^n} x \cdot x^{2n} =$$

$$\sum_{k=0}^{\infty} \frac{3 \cdot 5^n}{4^{n+1}} x^{2n+1}$$