Arclength Key

①

1\. a. $f'(x) = \frac{1}{2}x^2 - \frac{1}{2}x^{-2}$ $\quad [f'(x)]^2 = \frac{1}{4}x^4 - \frac{1}{2} + \frac{1}{2}x^{-4}$

$1 + [f'(x)]^2 = \frac{1}{4}x^4 + \frac{1}{2} + \frac{1}{2}x^{-4} = \left(\frac{1}{2}x^2 + \frac{1}{2}x^{-2}\right)^2$

$$\int_1^3 \sqrt{(1/2x^2 + \tfrac{1}{2}x^{-2})^2}\,dx = \int_1^3 \tfrac{1}{2}x^2 + \tfrac{1}{2}x^{-2}\,dx = \tfrac{1}{6}x^3 - \tfrac{1}{2x}\Big|_1^3 = \tfrac{14}{3}$$

b. $g'(x) = x^{1/2}$ $\quad [g'(x)] = x$ $\quad \int_8^{27} \sqrt{1+x}\,dx = \frac{2}{3}(1+x)^{3/2}\Big|_8^{27}$

$$= \tfrac{2}{3}\left[28^{3/2} - 27\right] \approx 7.05 \times 10^{12}$$

c. $h'(x) = \frac{1}{2}x^3 - \frac{1}{2}x^{-3}$ $\quad [h'(x)]^2 = \frac{1}{4}x^6 - \frac{1}{2} + \frac{1}{4}x^{-6}$

$1 + [h'(x)]^2 = \frac{1}{4}x^6 + \frac{1}{2} + \frac{1}{4}x^{-6} = \left(\frac{1}{2}x^3 + \frac{1}{2}x^{-3}\right)^2$

$$\int_1^4 \sqrt{(\tfrac{1}{2}x^3 + \tfrac{1}{2}x^{-3})^2}\,dx = \int_1^4 \tfrac{1}{2}x^3 + \tfrac{1}{2}x^{-3}\,dx = \tfrac{1}{8}x^4 - \tfrac{1}{4}x^{-2}\Big|_1^4 = \tfrac{2055}{64}$$

d. $k'(x) = \cot x$ $\quad [k'(x)]^2 + 1 = 1 + \cot^2 x = \csc^2 x$

$$\int_{\pi/4}^{3\pi/4} |\csc x|\,dx = \ln|\csc x + \cot x|\Big|_{\pi/4}^{3\pi/4}$$

$$= \ln|\sqrt{2} - 1| + \ln|\sqrt{2} - 1| \approx 1.76$$

e. $a'(x) = \sinh x$ $\quad \int_0^2 \sqrt{1 + \sinh^2 x}\,dx = \int_0^2 \sqrt{\cosh^2 x}\,dx =$

$$\int_0^2 \cosh x\,dx = \sinh x\Big|_0^2 = \sinh(2)$$

$$= \frac{e^2 - e^{-2}}{2}$$

f. $g(x) = \ln(e^x+1) - \ln(e^x-1)$

$$g'(x) = \frac{e^x}{e^x+1} - \frac{e^x}{e^x-1} = \frac{\cancel{e^{2x}} - e^x - \cancel{e^{2x}} - e^x}{e^{2x}-1} = \frac{-2e^x}{e^{2x}-1}$$

$$1+[g'(x)]^2 = \frac{4e^{2x}}{(e^{2x}-1)^2} + 1 = \frac{4e^{2x} + e^{4x} - 2e^{2x} + 1}{(e^{2x}-1)^2} = \frac{e^{4x}+2e^{2x}+1}{(e^{2x}-1)^2}$$

$$= \frac{(e^{2x}+1)^2}{(e^{2x}-1)^2}$$

$$\int_{\ln 2}^{\ln 3} \sqrt{\left(\frac{e^{2x}+1}{e^{2x}-1}\right)^2}\, dx = \int_{\ln 2}^{\ln 3} \frac{e^{2x}+1}{e^{2x}-1}\, dx \quad \frac{e^{-x}}{e^{-x}} =$$

$$\int_{\ln 2}^{\ln 3} \frac{e^x + e^{-x}}{e^x - e^{-x}}\, dx = \int_{\ln 2}^{\ln 3} \coth x \, dx = \ln|\sinh x| \Big|_{\ln 2}^{\ln 3}$$

$$= \ln\left(\frac{e^x - e^{-x}}{2}\right)\Big|_{\ln 2}^{\ln 3} = \ln\left(\frac{3-\frac{1}{3}}{2}\right) - \ln\left(\frac{2-\frac{1}{2}}{2}\right)$$

$$\approx .57536\ldots$$

2. a. $u'(y) = \frac{1}{2}(y^2+2)^{1/2} \cdot 2y =$

$y\sqrt{y^2+2}$

$$1 + [u'(y)]^2 = 1 + y^2(y^2+2) = y^4 + 2y^2 + 1 = (y^2+1)^2$$

$$\int_0^4 \sqrt{(y^2+1)^2}\, dy = \int_0^4 y^2+1 \, dy = \frac{1}{3}y^3 + y \Big|_0^4 = \frac{64}{3} + 4 = \frac{76}{3}$$

b. $v(y) = \frac{1}{3}(y^{3/2} - 3\sqrt{y}) = \frac{1}{3}y^{3/2} - y^{1/2}$

$v'(y) = \frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1} \qquad (v'(y))^2 = \frac{1}{4}y - \frac{1}{2} + \frac{1}{4}y^{-2}$

$$1 + [v'(y)]^2 = \frac{1}{4}y + \frac{1}{2} + \frac{1}{4}y^{-2} = \left(\frac{1}{2}y^{1/2} + \frac{1}{2}y^{-1/2}\right)^2$$

$$\int_1^4 \sqrt{\left(\frac{1}{2}y^{1/2} + \frac{1}{2}y^{-1/2}\right)^2}\, dy = \int_1^4 \frac{1}{2}y^{1/2} + \frac{1}{2}y^{-1/2}\, dy = \frac{1}{3}y^{3/2} + y^{1/2}\Big|_1^4 = \frac{8}{3} + 2 - \frac{1}{3} - 1$$

$$= 10/3$$

3. a. $y'(x) = -2x \qquad 1 + [y'(x)]^2 = 1 + 4x^2$

$\int_0^4 \sqrt{1+4x^2}\,dx \qquad 2x = \tan\theta \qquad \sqrt{1+4x^2} = \sec\theta$

$2dx = \sec^2\theta\, d\theta$

$\frac{1}{2}\int \sec\theta \cdot \sec^2\theta\, d\theta \qquad u = \sec\theta \qquad dv = \sec^2\theta$

$du = \sec\theta \tan\theta\, d\theta \qquad v = \tan\theta$

$\frac{1}{4}\ln\left|\sqrt{4x^2+1} + 2x\right| + \frac{1}{2}x\sqrt{4x^2+1}\,\Big|_0^4 = \frac{\ln(\sqrt{65}+8)}{4} + 2\sqrt{65}$

b. $r'(x) = -\frac{1}{x^2} \qquad 1 + [r'(x)]^2 = 1 + \frac{1}{x^4} = \frac{1+x^4}{x^4}$

$\int_1^5 \frac{\sqrt{1+x^4}}{x^2}\,dx\,\frac{x}{x} \qquad u = x^2 = \tan\theta \qquad \sqrt{1+x^4} = \sec\theta$

$\cancel{2}x\,dx = \frac{1}{2}\sec^2\theta\, d\theta$

$\frac{1}{2}\int \frac{\sec\theta \cdot \sec^2\theta\, d\theta}{\tan^{3/2}\theta}$ must do numerically

≈ 4.15145

c. $s'(x) = \cos x \qquad 1 + [s'(x)]^2 = 1 + \cos^2 x$

$\int_0^\pi \sqrt{1+\cos^2 x}\,dx$ (numerically) ≈ 3.8202

d. $w'(x) = -e^{-x} \qquad \int_0^2 \sqrt{1+e^{-2x}}\,dx \approx 2.22142$

e. $p'(x) = \frac{1}{1+x^2} \qquad \int_0^1 \sqrt{1+\left(\frac{1}{1+x^2}\right)^2}\,dx \approx 1.27798$

4a. $x^2+y^2=9 \Rightarrow y = \pm\sqrt{9-x^2} \qquad y' = \pm\frac{1}{2}(9-x^2)^{-1/2}\cdot -2x$

$= \frac{x}{\sqrt{9-x^2}}$

4a. cont'd. $1+(y')^2 = 1+\frac{x^2}{9-x^2} = \frac{9-\cancel{x^2}+\cancel{x^2}}{9-x^2} = \frac{9}{9-x^2}$

$\int_{-3}^{3}\sqrt{\frac{9}{9-x^2}} = \int_{-3}^{3}\frac{3}{\sqrt{9-x^2}}\,dx = 3\arcsin\left(\frac{x}{3}\right)\Big|_{-3}^{3} = 3[\pi-(-\pi)]$

$= 6\pi$

top half

$6\pi * 2 = \boxed{12\pi}$

4b. $y^{2/3} = 4-x^{2/3}$ $\quad y^2 = (4-x^{2/3})^3$ $\quad y = \pm(4-x^{2/3})^{3/2}$

$y' = \pm\frac{3}{2}(4-x^{2/3})^{1/2}\cdot -\frac{2}{3}x^{1/2} = \mp\sqrt{x}\,(4-x^{2/3})^{1/2}$

$1+(y')^2 = 1+x(4-x^{2/3}) = 1+4x-x^{5/3}$

$4\int_0^2\sqrt{1+4x-x^{5/3}}\,dx \approx 3.8276 * 4 = 15.3104$

5a. $y' = 1$ $\quad \int_0^1\sqrt{2}\,dx = \sqrt{2} \approx 1.41$

b. $y' = 2x$ $\quad \int_0^1\sqrt{1+4x^2}\,dx \approx 1.47894$

c. $y' = 3x^2$ $\quad \int_0^1\sqrt{1+9x^4}\,dx \approx 1.54787$

d. $y' = 5x^4$ $\quad \int_0^1\sqrt{1+25x^8}\,dx \approx 1.64056$

e. $y' = \frac{1}{2}x^{-1/2}$ $\quad \int_0^1\sqrt{1+\frac{1}{4x^2}}\,dx \approx$ breaks calculator at 0

Solve for $x = f(y)$ & redo

$x' = 2y$ $\quad \int_0^1\sqrt{1+4y^2}\,dy \approx 1.47894$

do the same for f & g

f. $x = y^{3/2}$ $\quad x' = \frac{3}{2}y^{1/2}$ $\quad \int_0^1\sqrt{1+\frac{9}{4}y}\,dy = 1.43971$

g. $x = y^4$ $\quad x' = 4y^3$ $\quad \int_0^1\sqrt{1+16y^6}\,dy \approx 1.60023$

Shortest is $y = x$ others are longer w/ higher/lower powers & symmetric in x/y