Math 2568 Proof Set 3

1. Let $T: V \to W$, x, y in V, and $T(x), T(y)$ in W.

1) since $T(\vec{0}) = \vec{0}$ since T is a linear transformation $\vec{0}$ is in W.

2) for $\vec{x}, \vec{y} \in V$, $\vec{x}+\vec{y} \in V$ since V is a vector space.
$T(\vec{x}) + T(\vec{y}) = T(\vec{x}+\vec{y})$ & since $\vec{x}, \vec{y}, \vec{x}+\vec{y}$ in V, by definition $T(\vec{x}), T(\vec{y}), T(\vec{x}+\vec{y})$ in W.

3) for $\vec{x} \in V$, $k\vec{x}$ in V since V is a vector space & since
$T(k\vec{x}) = kT(\vec{x})$, $kT(\vec{x})$ in W since $T(k\vec{x})$ in W.

Therefore W is a vector space.

2. Since $\sin 2t = 2 \sin t \cos t$ we can write one element in the set as a linear combination of the other, and so therefore the set is not linearly independent, & so it cannot be a basis.

To form a basis, eliminate either $\sin 2t$ or $\sin t \cos t$.

$B = \{\sin t, \sin 2t\}$

3. $p_1 = 1 + t^2 \approx \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\quad p_2 = t - 3t^2 \approx \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$ $\quad p_3 = 1 + t - 3t^2 \approx \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -3 & -3 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Therefore the vectors span $\mathbb{P}_2$ and are linearly independent. Thus, it is a basis for $\mathbb{P}_2$

4. Consider the first 2 standard basis elements for $\mathbb{P}_n$ $\{1, t\}$. These elements are linearly independent, but do not span the space. Adding the next basis element t^2, we get $\{1, t, t^2\}$. These are linearly independent but do not span $\mathbb{P}$. We continue in this way to $\mathbb{P}_n$ with basis elements $\{1, t, t^2, \ldots t^n\}$. These are independent but do not span the space of all polynomials since t^{n+1} is not a linear combination of $\{1, t, t^2, \ldots, t^n\}$. Therefore there is no highest degree polynomial and no finite basis for $\mathbb{P}$.

5. show m, p, q for instance.
any basis of $\mathbb{R}^n$ written as a matrix will row reduce to the identity since the basis must be both linearly independent and span $\mathbb{R}^n$. The identity has n pivots which implies that rank $A = n$. Since Rank A + dim Nul $A = n$ $\quad n +$ dim Nul $A = n \Rightarrow$ dim Nul $A = 0$. for this to be true, the only element of the Nullspace must be $\{0\}$. if $\{0\}$ is the only vector in the Nullspace this implies that the only solution to $A\vec{x} = \vec{0}$ is the trivial solution, which implies A is invertible. If A is invertible its columns are independent & span $\mathbb{R}^n$, so its columns form a basis for $\mathbb{R}^n$. Thus $m \Leftrightarrow p \Leftrightarrow q$.

6. $P\vec{q} = \vec{q} \;=\; (P-I)\vec{q} = \vec{0}$ Compare to the eigenvalue equation $(A-\lambda I)\vec{v} = \vec{0}$. This implies that $\lambda = 1$, and $\vec{v} = \vec{q}$ is the eigenvector for $\lambda = 1$ eigenvector.