Linear Algebra   Linear Transformations Key

1. $T: \vec{x} \in \mathbb{R}^3 \Rightarrow A\vec{x} \in \mathbb{R}^3$

$$A = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 2 \end{bmatrix} \qquad A\vec{x} = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 4x_2 + 0x_3 \\ 0x_1 + 0x_2 - x_3 \\ 0x_1 + 0x_2 + 2x_3 \end{bmatrix} = T$$

Since we can write T as a matrix, it satisfies the first two properties of linear transformations since all matrices have the property That $A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v}$ and $A(c\vec{u}) = c(A\vec{u})$, therefore so does T. Checking $T(0)=0$ we see that $\begin{bmatrix} 1 & 4 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ does equal $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. //

2. $T: \vec{x} \in \mathbb{R}^3 \rightarrow \vec{b} \in \mathbb{R}^3$

however since T can't be represented by a matrix, we check the properties by hand.

a) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$

consider $\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = \vec{u}$ and $\vec{v} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$

$$\vec{u}+\vec{v} = \begin{bmatrix} 1 \\ 7 \\ 3 \end{bmatrix} \qquad T\left(\begin{bmatrix} 1 \\ 7 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 3-7 \\ 49+5 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 54 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2-4 \\ 16+5 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 21 \\ 0 \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1-3 \\ 9+5 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 14 \\ 0 \end{bmatrix}$$

but $T(\vec{u}) + T(\vec{v}) = \begin{bmatrix} -4 \\ 35 \\ 0 \end{bmatrix}$ not $\begin{bmatrix} -4 \\ 54 \\ 0 \end{bmatrix}$

Since at least one property fails even once, This is not a linear transformation. //

3. This example is similar to The one variable derivative example, but ∇f, ∇g are vectors

3 con'td.

a) $\nabla f = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}$ $\quad \nabla g \begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix}$ $\quad \nabla f + \nabla g = \begin{bmatrix} f_x + g_x \\ f_y + g_y \\ f_z + g_z \end{bmatrix}$

$\nabla(f+g) = \begin{bmatrix} (f+g)_x \\ (f+g)_y \\ (f+g)_z \end{bmatrix}$ Since these are partial derivative operators w/ the same properties as $\frac{d}{dx}$ we can say that

$\begin{bmatrix} (f+g)_x \\ (f+g)_y \\ (f+g)_z \end{bmatrix} = \begin{bmatrix} f_x + g_x \\ f_y + g_y \\ f_z + g_z \end{bmatrix}$ Therefore $\nabla(f+g) = \nabla f + \nabla g$ ✓

b) $\nabla(kf) = \begin{bmatrix} (kf)_x \\ (kf)_y \\ (kf)_z \end{bmatrix}$ by properties of derivatives $= \begin{bmatrix} kf_x \\ kf_y \\ kf_z \end{bmatrix}$

which $= k \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = k \nabla f$ ✓

c) $\nabla(0) = \begin{bmatrix} \partial/\partial x\, [0] \\ \partial/\partial y\, [0] \\ \partial/\partial z\, [0] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ✓

Since ∇ satisfies all the properties of a linear transformation it is one. //

4. $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$ let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ w/ $u_1, u_2, v_1, v_2 \in \mathbb{R}$

$\vec{u} + \vec{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$ $\quad c\vec{u} = \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}$

a) $A(\vec{u} + v) = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix} = \begin{bmatrix} 2u_1 + 2v_1 + 3u_2 + 3v_2 \\ -u_1 - v_1 + 4u_2 + 4v_2 \end{bmatrix}$

$A(\vec{u}) + A(\vec{v}) = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2u_1 + 3u_2 \\ -u_1 + 4u_2 \end{bmatrix} + \begin{bmatrix} 2v_1 + 3v_2 \\ -v_1 + 4v_2 \end{bmatrix} =$

$\begin{bmatrix} 2u_1 + 2v_1 + 3u_2 + 3v_2 \\ -u_1 - v_1 + 4u_2 + 4v_2 \end{bmatrix}$ ✓

4 cont'd

b) $A(c\vec{u}) = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix} = \begin{bmatrix} 2cu_1 + 3cu_2 \\ -cu_1 + 4cu_2 \end{bmatrix}$

$c(A\vec{u}) = c\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = c\begin{bmatrix} 2u_1 + 3u_2 \\ -u_1 + 4u_2 \end{bmatrix} = \begin{bmatrix} c(2u_1 + 3u_2) \\ c(-u_1 + 4u_2) \end{bmatrix}$

which by the distributive property gives us $A(c\vec{u})$. ✓

c) $\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ✓

therefore, A is a linear transformation. Since A is a matrix and must satisfy properties of matrices, This was to be expected. //

5. $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ let $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $V_1 = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ w/ $u_1, u_2, v_1, v_2 \in \mathbb{R}$

a) $A(\vec{u}+\vec{v}) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix} = \begin{bmatrix} \cos\theta(u_1+v_1) + \sin\theta(u_2+v_2) \\ -\sin\theta(u_1+v_1) + \cos\theta(u_2+v_2) \end{bmatrix}$

$A\vec{u} + Av = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} =$

$\begin{bmatrix} \cos\theta \cdot u_1 + \sin\theta\, u_2 \\ -\sin\theta \cdot u_1 + \cos\theta\, u_2 \end{bmatrix} + \begin{bmatrix} \cos\theta\, v_1 + \sin\theta \cdot v_2 \\ -\sin\theta\, v_1 + \cos\theta\, v_2 \end{bmatrix} =$

$\begin{bmatrix} \cos\theta\, u_1 + \sin\theta \cdot u_2 + \cos\theta\, v_1 + \sin\theta \cdot v_2 \\ -\sin\theta \cdot u_1 + \cos\theta \cdot u_2 - \sin\theta\, v_1 + \cos\theta\, v_2 \end{bmatrix}$

Collecting $\sin\theta$ & $\cos\theta$ terms in each component gives

$\begin{bmatrix} \cos\theta(u_1+v_1) + \sin\theta(u_2+v_2) \\ -\sin\theta(u_1+v_1) + \cos\theta(u_2+v_2) \end{bmatrix}$ which was our previous result. ✓

b). $A(c\vec{u}) \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix} = \begin{bmatrix} cu_1\cos\theta + cu_2\sin\theta \\ -cu_1\sin\theta + cu_2\cos\theta \end{bmatrix} = c\begin{bmatrix} u_1\cos\theta + u_2\sin\theta \\ -u_1\sin\theta + u_2\cos\theta \end{bmatrix}$

$c(A\vec{u}) = c\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = c\begin{bmatrix} \cos\theta \cdot u_1 + \sin\theta\, u_2 \\ -\sin\theta \cdot u_1 + \cos\theta \cdot u_2 \end{bmatrix}$ which is the same as above.

c) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ✓ //

4

6. Consider the integration operator $\int_0^x \int_0^y f(u,v)\,du\,dv$ for generic functions $f(u,v)$ and $g(u,v)$.

a) $\int_0^x\int_0^y (f(u,v)+g(u,v))\,du\,dv = \int_0^x \int_0^y f(u,v)\,du\,dv + \int_0^x \int_0^y g(u,v)\,du\,dv$

by properties of the integration operator.

b) $\int_0^x \int_0^y k f(u,v)\,du\,dv = k \int_0^x \int_0^y f(u,v)\,du\,dv$ by properties of integration.

c) $\int_0^x \int_0^y 0\,du\,dv = 0$ by properties of definite integrals.

7. $A = \begin{bmatrix} -2 & 0 & 1 \\ 3 & -1 & 5 \\ 2 & -4 & 0 \end{bmatrix}$ since A is a matrix it satisfies the matrix properties $A(\vec{u}+v) = A\vec{u} + A\vec{v}$, the first of our properties of a linear transformation, and also $A(c\vec{u}) = c(A\vec{u})$, the second property. It's also clear that

$$\begin{bmatrix} -2 & 0 & 1 \\ 3 & -1 & 5 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad //$$

8. $T: \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \to \begin{bmatrix} r \\ \theta \end{bmatrix}$ in $\mathbb{R}^2$

this transformation is not linear since

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \to \begin{bmatrix} \sqrt{2} \\ \pi/4 \end{bmatrix}$ but $2\begin{bmatrix} 1 \\ 1 \end{bmatrix} \not\to \begin{bmatrix} 2\sqrt{2} \\ \pi/2 \end{bmatrix}$

rather, it maps $\begin{bmatrix} 2 \\ 2 \end{bmatrix} \to \begin{bmatrix} 2\sqrt{2} \\ \pi/4 \end{bmatrix}$ Therefore the second property (at least) fails. //