Math 2568 Homework #5 Key Supplement

3. k. true

l. false. if one eigenvalue is 0, then A will be diagonalizable but not invertible.

8. a. $\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$ $(1-\lambda)(-1-\lambda)-0 \Rightarrow \lambda_1=1,\ \lambda_2=-1$ diagonalizable

$\begin{bmatrix} 1-1 & 0 \\ 6 & -1-1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 6 & -2 \end{bmatrix}$ $6x_1 - 2x_2 = 0 \Rightarrow \begin{matrix} x_1 = \frac{1}{3}x_2 \\ x_2 = x_2 \end{matrix}$ $\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\begin{bmatrix} 1-(-1) & 0 \\ 6 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 6 & 0 \end{bmatrix}$ $\begin{matrix} 2x_1 = 0 \\ x_2 = x_2 \end{matrix} \Rightarrow \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

basis of eigenvectors for $\mathbb{R}^2$ $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $P = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

$A = PDP^{-1} \Rightarrow A^k = PD^kP^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (-1)^k \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$ $(1-\lambda)(2-\lambda) - 12 = \lambda^2 - 3\lambda + 2 - 12 = \lambda^2 - 3\lambda - 10 = 0$

$(\lambda-5)(\lambda+2) = 0$ $\lambda_1 = 5,\ \lambda_2 = -2$ diagonalizable

$\begin{bmatrix} 1-5 & 3 \\ 4 & 2-5 \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ 4 & -3 \end{bmatrix}$ $4x_1 + 3x_2 = 0$ $\begin{matrix} x_1 = \frac{3}{4}x_2 \\ x_2 = x_2 \end{matrix}$ $\vec{v}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$\begin{bmatrix} 1+2 & 3 \\ 4 & 2+2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix}$ $x_1 + x_2 = 0$ $\begin{matrix} x_1 = -x_2 \\ x_2 = x_2 \end{matrix}$ $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$D = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$ $P = \begin{bmatrix} 3 & -1 \\ 4 & 1 \end{bmatrix}$

$A^k = \begin{bmatrix} 3 & -1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 5^k & 0 \\ 0 & (-2)^k \end{bmatrix} \begin{bmatrix} 1/7 & 1/7 \\ -4/7 & 3/7 \end{bmatrix}$

c. $\begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}$ $\lambda_1 = 1, \lambda_2 = 5$

$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & -1 \\ -1 & -2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $x_1 + 2x_2 - x_3 = 0$

$\Rightarrow x_1 = -2x_2 + x_3$ $\vec{V}_{11} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

$x_2 = x_2$

$x_3 = \quad x_3$ $\vec{V}_{12} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} -3 & 2 & -1 \\ 1 & -2 & -1 \\ -1 & -2 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $x_1 + x_3 = 0$

$x_2 + x_3 = 0$

$\Rightarrow$ $x_1 = -x_3$

$x_2 = -x_3$ $\Rightarrow \vec{V}_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

$x_3 = x_3$

it is diagonalizable since eigenspace is $\mathbb{R}^3$

$$A^k = \begin{bmatrix} -2 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5^k \end{bmatrix} \begin{bmatrix} -1/4 & 1/2 & 1/4 \\ 1/4 & 1/2 & 3/4 \\ -1/4 & -1/2 & 1/4 \end{bmatrix}$$

d. $\begin{bmatrix} 2 & -2 & -2 \\ 3 & -3 & -2 \\ 2 & -2 & -2 \end{bmatrix}$ $\lambda_1 = -2, \lambda_2 = -1, \lambda_3 = 0$

$\begin{bmatrix} 4 & -2 & -2 \\ 2 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $x_1 - x_3 = 0$, $x_2 - x_3 = 0$ $\quad$ $x_1 = x_3$, $x_2 = x_3$, $x_3 = x_3$ $\vec{V}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 3 & -2 & -2 \\ 2 & -2 & -2 \\ 2 & -2 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$ $x_1 - x_3 = 0$, $x_2 - \frac{1}{2}x_3 = 0$ $\quad$ $x_1 = x_3$, $x_2 = \frac{1}{2}x_3$, $x_3 = x_3$ $\vec{V}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

$\begin{bmatrix} 2 & -2 & -2 \\ 3 & -3 & -2 \\ 2 & -2 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $x_1 - x_2 = 0$, $x_3 = 0$ $\quad$ $x_1 = x_2$, $x_2 = x_2$, $x_3 = 0$ $\vec{V}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

11 Cont'd

$$\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix} \Rightarrow 3x_1 + x_2 = 0 \Rightarrow \begin{array}{l} x_1 = -\frac{1}{3}x_2 \\ x_2 = x_2 \end{array} \quad \vec{v}_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix} \Rightarrow -x_1 + 2x_2 = 0 \quad \begin{array}{l} x_1 = 2x_2 \\ x_2 = x_2 \end{array} \quad v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}$$

9. A must be diagonalizable, since there are 3 unique eigenvalues each one must be associated w/ at least one eigenvector independent of the others. However, we are told that one of the eigenvalues has an eigenspace which is 2-dimensional thus we have $1+1+2=4$ linearly indep. vectors that span the space $\mathbb{R}^4$ and thus form a basis for the space. that is enough for diagonalizability.

10. $T: P_2 \to P_3$ $\quad p(t) = a_0 + a_1 t + a_2 t^2$

$$(t+3)p(t) = (t-3)(a_0 + a_1 t + a_2 t^2) =$$

$$a_0 t + a_1 t^2 + a_2 t^3 - 3a_0 - 3a_1 t - 3a_2 t^2$$

$$(-3a_0) + (a_0 - 3a_1)t + (a_1 - 3a_2)t^2 + (a_2)t^3$$

$$T = \begin{bmatrix} -3a_0 \\ a_0 - 3a_1 \\ a_1 - 3a_2 \\ a_2 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 1 & -3 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$p(t) = 3 - 2t + t^2 = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \Rightarrow A\,p(t) = \begin{bmatrix} -3 & 0 & 0 \\ 1 & -3 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -9 \\ 9 \\ -5 \\ 1 \end{bmatrix}$$

it can be written as a matrix, so yes, it's linear

A is the matrix in question.

11. $A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$ $\quad \begin{vmatrix} 1-\lambda & 2 \\ 3 & -4-\lambda \end{vmatrix} = (1-\lambda)(-4-\lambda) - 6 = -4 - \lambda + 4\lambda + \lambda^2 - 6$

$$= \lambda^2 + 3\lambda - 10$$
$$= (\lambda+5)(\lambda-2)$$
$$\lambda = -5, 2$$

$[T]_B = \begin{bmatrix} -5 & 0 \\ 0 & 2 \end{bmatrix}$ basis of eigenvectors $B = \left\{ \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$