Math 2255 Laplace transforms Key

a. i. $\mathcal{L}\{e^{-3t}\} = \int_0^\infty e^{-st}e^{-3t}\,dt = \int_0^\infty e^{-(s+3)t}\,dt = \left.\dfrac{e^{-(s+3)t}}{-(s+3)}\right|_0^\infty$

$= \dfrac{-1}{s+3}\left[0-1\right] = \dfrac{1}{s+3}$

ii. $\mathcal{L}\{e^{-at}\} = \int_0^\infty e^{-st}e^{-at}\,dt = \int_0^\infty e^{-(s+a)t}\,dt = \left.\dfrac{e^{-(s+a)t}}{-(s+a)}\right|_0^\infty$

$= \dfrac{-1}{s+a}\left[0-1\right] = \dfrac{1}{s+a}$

iii. $\mathcal{L}\{\sin(t)\} = \int_0^\infty e^{-st}\sin t\,dt = -\frac{1}{s}e^{-st}\sin t + \frac{1}{s}\int_0^\infty e^{-st}\cos t\,dt$

$u = \sin t \quad dv = e^{-st}dt$
$du = \cos t\,dt \quad v = -\frac{1}{s}e^{-st}$

$u = \cos t \quad dv = e^{-st}dt$
$du = \sin t\,dt \quad v = -\frac{1}{s}e^{-st}$

$= -\frac{1}{s}e^{-st}\sin t + \frac{1}{s}\left[-\frac{1}{s}e^{-st}\cos t - \frac{1}{s}\int_0^\infty e^{-st}\sin t\,dt\right] = \int_0^\infty e^{-st}\sin t\,dt$

$-\frac{1}{s}e^{-st}\sin t - \frac{1}{s^2}e^{-st}\cos t - \cancel{\frac{1}{s^2}\int_0^\infty e^{-st}\sin t\,dt} = \int_0^\infty e^{-st}\sin t\,dt$
$\qquad + \cancel{\frac{1}{s^2}\int_0^\infty e^{-st}\sin t\,dt} \quad + \frac{1}{s^2}\int_0^\infty e^{-st}\sin t\,dt$

$-\frac{1}{s}e^{-st}\sin t - \frac{1}{s^2}e^{-st}\cos t = \underbrace{\left(1+\frac{1}{s^2}\right)}_{=\frac{s^2+1}{s^2}}\int_0^\infty e^{-st}\sin t\,dt$

$\dfrac{s^2}{s^2+1}\left[-\frac{1}{s}e^{-st}\sin t - \frac{1}{s^2}e^{-st}\cos t\right]_0^\infty = \int_0^\infty e^{-st}\sin t\,dt$

$\dfrac{s^2}{s^2+1}\left[0-0+0+\frac{1}{s^2}(1)(1)\right] = \dfrac{1}{s^2+1}$

iv. $\mathcal{L}\{\sin(2t)\} = \int_0^\infty e^{-st}\sin 2t\,dt = -\frac{1}{s}e^{-st}\sin 2t + \frac{2}{s}\int e^{-st}\cos 2t\,dt$

$u = \sin 2t \quad dv = e^{-st}dt$
$du = 2\cos 2t\,dt \quad v = -\frac{1}{s}e^{-st}$

$u = \cos 2t \quad dv = e^{-st}dt$
$du = -2\sin 2t\,dt \quad v = -\frac{1}{s}e^{-st}$

IV. cont'd

$$= -\frac{1}{s}e^{-st}\sin 2t + \frac{2}{s}\left[-\frac{1}{s}e^{-st}\cos 2t - \frac{2}{s}\int_0^\infty e^{-st}\sin 2t\,dt\right] = \int_0^\infty e^{-st}\sin 2t\,dt$$

$$-\frac{1}{s}e^{-st}\sin 2t - \frac{2}{s^2}e^{-st}\cos 2t - \cancel{\frac{4}{s^2}\int_0^\infty e^{-st}\sin 2t\,dt} = \int_0^\infty e^{-st}\sin 2t\,dt$$
$$+\cancel{\frac{4}{s^2}\int_0^\infty e^{-st}\sin 2t\,dt} + \frac{4}{s^2}\int_0^\infty e^{-st}\sin 2t\,dt$$

$$-\frac{1}{s}e^{-st}\sin 2t - \frac{2}{s^2}e^{-st}\cos 2t\Big|_0^\infty = \underbrace{\left(1+\frac{4}{s^2}\right)}_{\frac{s^2+4}{s^2}}\int_0^\infty e^{-st}\sin 2t\,dt$$

$$\frac{s^2}{s^2+4}\left[-\frac{1}{s}e^{-st}\sin 2t - \frac{2}{s^2}e^{-st}\cos 2t\right]_0^\infty$$

$$\frac{s^2}{s^2+4}\left[0 - 0 + 0 + \frac{2}{s^2}\right] = \frac{2}{s^2+4}$$

V. $\mathcal{L}\{\cos(t)\} = \int_0^\infty e^{-st}\cos t\,dt = -\frac{1}{s}e^{-st}\cos t - \frac{1}{s}\int_0^\infty e^{-st}\sin t\,dt =$

$u = \cos t \quad dv = e^{-st}dt$ $\qquad u = \sin t \quad dv = e^{-st}dt$

$du = -\sin t\,dt \quad v = -\frac{1}{s}e^{-st}$ $\qquad du = \cos t\,dt \quad v = -\frac{1}{s}e^{-st}$

$$-\frac{1}{s}e^{-st}\cos t - \frac{1}{s}\left[-\frac{1}{s}e^{-st}\sin t + \frac{1}{s}\int_0^\infty e^{-st}\cos t\,dt\right] = \int_0^\infty e^{-st}\cos t\,dt$$

$$-\frac{1}{s}e^{-st}\cos t + \frac{1}{s^2}e^{-st}\sin t\Big|_0^\infty - \cancel{\frac{1}{s^2}\int_0^\infty e^{-st}\cos t\,dt} = \int_0^\infty e^{-st}\cos t\,dt$$
$$+\cancel{\frac{1}{s^2}\int_0^\infty e^{-st}\cos t\,dt} \quad \frac{1}{s^2}\int_0^\infty e^{-st}\cos t\,dt$$

$$\underbrace{\left(1+\frac{1}{s^2}\right)}_{\left(\frac{s^2+1}{s^2}\right)}\int_0^\infty e^{-st}\cos t\,dt$$

$$\frac{s^2}{s^2+1}\left[-\frac{1}{s}e^{-st}\cos t + \frac{1}{s^2}e^{-st}\sin t\right]_0^\infty = \frac{s^2}{s^2+1}\left[0 + 0 + \frac{1}{s} - 0\right] = \frac{s}{s^2+1}$$

VI. $\mathcal{L}\{\sinh(t)\} = \int_0^\infty e^{-st}\sinh t\,dt = \frac{1}{2}\int_0^\infty e^{-st}e^{t} - e^{-st}e^{-t}\,dt =$

$$\frac{1}{2}\int_0^\infty e^{-(s-1)t} - e^{-(s+1)t}\,dt = \frac{1}{2}\left[\frac{-1}{s-1}e^{-(s-1)t} + \frac{1}{s+1}e^{-(s+1)t}\right]_0^\infty$$

$$= \frac{1}{2}\left[0 + 0 + \frac{1}{s-1} - \frac{1}{s+1}\right] = \frac{1}{2}\left[\frac{s+1-(s-1)}{(s-1)(s+1)}\right] = \frac{1}{2}\left[\frac{2}{s^2-1}\right] = \frac{1}{s^2-1}$$

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a. viii $\mathcal{L}\{f(t)\}$ for $f(t) = \begin{cases} 0 & 0 \le t \le 1 \\ 1 & t \ge 1 \end{cases}$

$$= \int_0^\infty e^{-st} f(t)\,dt = \int_0^1 e^{-st}(0)\,dt + \int_1^\infty e^{-st}\,1\,dt = \int_1^\infty e^{-st}\,dt =$$

$$-\frac{1}{s}e^{-st}\Big|_1^\infty = 0 + \frac{1}{s}e^{-s(1)} = \frac{1}{s}e^{-s}$$

viii. $\mathcal{L}\{t^3\} = \int_0^\infty e^{-st} t^3\,dt =$

+/-	u	dv
+	t^3	e^{-st}
-	$3t^2$	$-\frac{1}{s}e^{-st}$
+	$6t$	$\frac{1}{s^2}e^{-st}$
-	6	$-\frac{1}{s^3}e^{-st}$
+	0	$\frac{1}{s^4}e^{-st}$

$$-\frac{1}{s}t^3e^{-st} - \frac{3}{s^2}t^2e^{-st} - \frac{6}{s^3}te^{-st} - \frac{6}{s^4}e^{-st}\Big|_0^\infty$$

$$= 0 - 0 - 0 - 0 + 0 + 0 + 0 + \frac{6}{s^4}$$

$$= \frac{6}{s^4}$$

b. i. $\mathcal{L}\{e^{t+5}\} = \int_0^\infty e^{-st}e^{t+5}\,dt = e^5\int_0^\infty e^{-(s-1)t}\,dt = e^5\left[\frac{-1}{s-1}e^{-(s-1)t}\right]_0^\infty$

$$e^5\left[0 + \frac{1}{s-1}\right] = \frac{e^5}{s-1}$$

ii. $\mathcal{L}\{2t^2 - 3t + 4\} = \int_0^\infty e^{-st}(2t^2 - 3t + 4)\,dt$

+/-	u	dv
+	$2t^2-3t-4$	e^{-st}
-	$4t-3$	$\frac{-1}{s}e^{-st}$
+	4	$\frac{1}{s^2}e^{-st}$
-	0	$-\frac{1}{s^3}e^{-st}$

$$-\frac{1}{s}(2t^2-3t-4)e^{-st} - \frac{1}{s^2}(4t-3)e^{-st} - \frac{4}{s^3}e^{-st}\Big|_0^\infty$$

$$0 - 0 - 0 + \frac{1}{s}(-4) + \frac{1}{s^2}(-3) + \frac{4}{s^3}$$

$$= \frac{-4}{s} - \frac{3}{s^2} + \frac{4}{s^3} = \frac{-4s^2 - 3s + 4}{s^3}$$

iii. $\mathcal{L}\{t \sin t\} = \int_0^\infty e^{-st}\,t \sin t\,dt$

$u = t \qquad dv = e^{-st}\sin t$

$du = dt \qquad v = \int e^{-st}\sin t = \frac{s^2}{s^2+1}\left[-\frac{1}{s}e^{-st}\sin t - \frac{1}{s^2}e^{-st}\cos t\right]$

from a iii.

$$= \frac{s^2t}{s^2+1}\left[-\frac{1}{s}e^{-st}\sin t - \frac{1}{s^2}e^{-st}\cos t\right]_0^\infty + \frac{s^2}{s^2+1}\int_0^\infty +\frac{1}{s}e^{-st}\sin t + \frac{1}{s^2}e^{-st}\cos t\,dt$$

$$= \frac{s^2t}{s^2+1}\left[-\frac{1}{s}e^{-st}\sin t - \frac{1}{s^2}e^{-st}\cos t\right]_0^\infty + \frac{s^2}{s^2+1}\left\{\frac{1}{s}\left[-\frac{1}{s}e^{-st}\sin t - \frac{1}{s^2}e^{-st}\cos t\right]\frac{s^2}{s^2+1} + \frac{1}{s^2+1}\left[\frac{1}{s}e^{-st}\cos t + \frac{1}{s^2}e^{-st}\sin t\right]\right\}_0^\infty$$

from a. v.

b iii contd

$$\frac{-st}{s^2+1}e^{-st}\sin t - \frac{t}{s^2+1}e^{-st}\cos t\Big|_0^\infty + \frac{s^2}{s^2+1}\left\{-\frac{1}{s^2}e^{-st}\sin t - \frac{1}{s^3}e^{-st}\cos t\right\}\left(\frac{s^2}{s^2+1}\right)\Big|_0^\infty$$

$$[0-0+0+0] + \left(\frac{s^2}{s^2+1}\right)^2\left[0-0+0+\frac{1}{s^3}(1)(1)\right] +$$

$$+\left(\frac{s^2}{s^2+1}\right)\left(\frac{1}{s^2+1}\right)\left[-\frac{1}{s}e^{-st}\cos t + \frac{1}{s^2}e^{-st}\sin t\right]_0^\infty$$

$$\frac{s^2}{(s^2+1)^2}\left[0+0+\frac{1}{s}(1)(1)-0\right]$$

$$= \frac{s^4}{(s^2+1)^2}\cdot\frac{1}{s^3} + \frac{s^2}{(s^2+1)^2}\cdot\frac{1}{s} = \frac{s}{(s^2+1)^2} + \frac{s}{(s^2+1)^2} = \frac{2s}{(s^2+1)^2}$$

iv. $\mathcal{L}\{e^t\sin t\} = \int_0^\infty e^{-st}e^t\sin t\,dt = \int_0^\infty e^{-(s-1)t}\sin t\,dt$

compare w/ a.iii

$$= \frac{(s-1)^2}{(s-1)^2+1}\left[-\frac{1}{s-1}e^{-(s-1)t}\sin t - \frac{1}{(s-1)^2}e^{-(s-1)t}\cos t\right]_0^\infty =$$

$$\frac{(s-1)^2}{(s-1)^2+1}\left[0-0+0+\frac{1}{(s-1)^2}(1)(1)\right] = \frac{1}{(s-1)^2+1}$$

v. $\mathcal{L}\{e^t\cosh t\} = \int_0^\infty e^{-st}e^t\cosh t = \frac{1}{2}\int_0^\infty e^{-st}e^t(e^t+e^{-t})\,dt$

$$= \frac{1}{2}\int_0^\infty e^{-st}(e^{2t}+1)\,dt = \frac{1}{2}\int_0^\infty e^{-(s-2)t} + e^{-st}\,dt =$$

$$\frac{1}{2}\left[-\frac{1}{s-2}e^{-(s-2)t} - \frac{1}{s}e^{-st}\right]_0^\infty = \frac{1}{2}\left[0-0+\frac{1}{s-2}(1)+\frac{1}{s}(1)\right] =$$

$$\frac{1}{2}\left[\frac{s+s-2}{s(s-2)}\right] = \frac{1}{2}\left[\frac{2s-2}{s^2-2s}\right] = \left[\frac{s-1}{s^2-2s}\right] = \frac{s-1}{s^2-2s+1-1} = \frac{s-1}{(s-1)^2-1}$$

(5)

vi. $\mathcal{L}\{\cos^2 t\} = \mathcal{L}\{\frac{1}{2}(1+\cos 2t)\} =$

$\frac{1}{2}[\underbrace{\mathcal{L}\{1\}}_{\text{from ex. 1}} + \mathcal{L}\{\cos 2t\}] = \frac{1}{2}\int_0^\infty e^{-st} + \frac{1}{2}\int_0^\infty e^{-st}\cos 2t\,dt$

by analogy w/ a.v and a.iv.

$\frac{1}{2}[\frac{1}{s} + \frac{2s}{s^2+4}] = \frac{1}{2s} + \frac{s}{s^2+4}$

c.i. $\mathcal{L}\{e^{t+5}\} = \mathcal{L}\{e^t e^5\} = e^5\mathcal{L}\{e^t\}$ $a=1$

$= e^5 \cdot \frac{1}{s-1} = \frac{e^5}{s-1}$

ii. $\mathcal{L}\{(3t-1)^2\} = \mathcal{L}\{9t^2-6t+1\} = 9\mathcal{L}\{t^2\} - 6\mathcal{L}\{t\}$

$+\mathcal{L}\{1\} = 9(\frac{2}{s^3}) - 6(\frac{1}{s^2}) + \frac{1}{s} = \frac{18}{s^3} - \frac{6}{s^2} + \frac{1}{s}$

$= \frac{18 - 6s + s^2}{s^3}$

iii. $\mathcal{L}\{\sin(t)\cos(t)\} = \mathcal{L}\{\frac{1}{2}\sin 2t\} = \frac{1}{2}\mathcal{L}\{\sin 2t\} =$

$k=2$

$\frac{1}{2}\cdot\frac{2}{s^2+4} = \frac{1}{s^2+4}$

iv. $\mathcal{L}\{e^{-t}\cosh t\}$ using $\cosh t = \frac{e^t+e^{-t}}{2} =$

$\frac{1}{2}\mathcal{L}\{e^{-t}(e^t+e^{-t})\} = \frac{1}{2}\mathcal{L}\{(1+e^{-2t})\} =$

$\frac{1}{2}[\mathcal{L}\{1\} + \mathcal{L}\{e^{-2t}\}] = \frac{1}{2}[\frac{1}{s} + \frac{1}{s+2}] = \frac{1}{2}[\frac{s+2+s}{s(s+2)}] =$

$a=-2$

$\frac{1}{2}[\frac{2s+2}{s^2+2s}] = \frac{s+1}{(s^2+2s+1)-1} = \frac{s+1}{(s+1)^2-1}$

or using $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$ $a=-1$

$\mathcal{L}\{\cosh t\} = \frac{s}{s^2-1} = F(s) \Rightarrow F(s+1) = \frac{s+1}{(s+1)^2-1}$

c. v. $\mathcal{L}\{\cos^2(t)\} = \mathcal{L}\{\frac{1}{2}(1+\cos 2\theta)\} = \frac{1}{2}\mathcal{L}\{1+\cos 2\theta\}$ (6)

$= \frac{1}{2}[\mathcal{L}\{1\} + \underset{k=2}{\mathcal{L}\{\cos 2\theta\}} = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right] = \frac{1}{2}\left[\frac{s^2+4+s}{s(s^2+4)}\right] =$

$\frac{s^2+s+4}{2s(s^2+4)}$

vi. $\mathcal{L}\{12t^5\} = 12\mathcal{L}\{t^5\} = 12 \cdot \frac{5!}{s^6} = 12 \cdot \frac{120}{s^6} = \frac{1440}{s^6}$

vii. $\mathcal{L}\{f(t)\} = \begin{cases} 0 & 0 \le t \le 3 \\ 7 & t \ge 3 \end{cases} = 7\begin{cases} 0 & 0 \le t \le 3 \\ 1 & t > 3 \end{cases}$

unit step
breaks at $3 = a$

$= 7\mathcal{L}\{u(t-3)\} = 7\frac{e^{-3s}}{s} = \frac{7e^{-3s}}{s}$

d. i. $\mathcal{L}^{-1}\{(\frac{2}{s} - \frac{1}{s^3})^2\} = \mathcal{L}^{-1}\{\frac{4}{s^2} - \frac{4}{s^4} + \frac{1}{s^6}\} =$

compare $\frac{1!}{s^2}$ $\quad \frac{3! = 6}{s^4}$ $\quad \frac{5!}{s^6}$

$4\underset{n=1}{\mathcal{L}^{-1}\{\frac{1!}{s^2}\}} - \frac{4}{6}\underset{n=3}{\mathcal{L}^{-1}\{\frac{3!}{s^4}\}} + \frac{1}{120}\underset{n=5}{\mathcal{L}^{-1}\{\frac{5!}{s^6}\}} =$

$4t - \frac{2}{3}t^3 + \frac{1}{120}t^5$

ii. $\mathcal{L}^{-1}\{\frac{1}{5s-2}\} = \frac{1}{5}\mathcal{L}^{-1}\{\frac{1}{s-2/5}\} = \frac{1}{5}e^{2/5 t}$

$a = 2/5$

iii. $\mathcal{L}^{-1}\{\frac{4s}{4s^2+1}\} = \frac{4}{4}\mathcal{L}^{-1}\{\frac{s}{s^2+\frac{1}{4}}\} = \cos\frac{1}{2}t$

$k = \frac{1}{2}$

iv. $\mathcal{L}^{-1}\{\frac{2s-6}{s^2+9}\} = 2\mathcal{L}^{-1}\{\frac{s}{s^2+9}\} - 2\mathcal{L}^{-1}\{\frac{3}{s^2+9}\}$

$k=3 \qquad k=3$

$2\cos 3t - 2\sin 3t$

d. v. $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2-4s}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{s(s-4)}\right\} = \mathcal{L}^{-1}\left\{\frac{A}{s} + \frac{B}{s-4}\right\}$

$= A \cdot 1 + Be^{4t} \Rightarrow -\frac{1}{4} + \frac{5}{4}e^{4t}$

$A(s-4) + B(s) = s+1$

$s=0$
$-4A = 1 \Rightarrow A = -\frac{1}{4}$

$s=4$
$4B = 5 \Rightarrow B = \frac{5}{4}$

vi. $\mathcal{L}^{-1}\left\{\frac{s-3}{s^2-3}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2-3}\right\} - \sqrt{3}\,\mathcal{L}^{-1}\left\{\frac{\sqrt{3}}{s^2-3}\right\}$

$k=\sqrt{3}$ $\quad$ $k=\sqrt{3}$

$= \cosh\sqrt{3}t - \sqrt{3}\sinh\sqrt{3}t$

vii. $\mathcal{L}^{-1}\left\{\frac{s}{(s-2)(s-3)(s-6)}\right\} = \mathcal{L}^{-1}\left\{\frac{A}{s-2} + \frac{B}{s-3} + \frac{C}{s-6}\right\} =$

$a=2$ $\quad$ $a=3$ $\quad$ $a=6$

$Ae^{2t} + Be^{3t} + Ce^{6t} \Rightarrow \frac{1}{2}e^{2t} - e^{3t} + \frac{1}{2}e^{6t}$

$A(s-3)(s-6) + B(s-2)(s-6) + C(s-2)(s-3) = s$

$s=2$
$A(-1)(-4) = 2 \Rightarrow 4A = 2 \Rightarrow A = \frac{1}{2}$

$s=3$
$B(1)(-3) = 3 \Rightarrow -3B = 3 \Rightarrow B = -1$

$s=6$
$C(4)(3) = 6 \Rightarrow 12C = 6 \Rightarrow C = \frac{1}{2}$

viii. $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}\right\} =$

$k=1$ $\quad$ $k=2$

$A\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + B\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + C\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{D}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$

$= A\cos t + B\sin t + C\cos 2t + \frac{D}{2}\sin 2t$

$As(s^2+4) + B(s^2+4) + Cs(s^2+1) + D(s^2+1) = 1$

$s=0 \quad 4B + D = 1$ $\qquad$ $s=1 \quad 5A + 5B + 2C + 2D = 1$

$s=2 \quad 16A + 8B + 6C + 3D = 1$ $\qquad$ $s=-1 \quad -5A + 5B - 2C + 2D = 1$

dviii cont'd

$$5\cancel{A}+5B+2\cancel{C}+2D=1$$
$$-\cancel{8}A+5B-\cancel{2}C+2D=1$$

$10B+4D=2 \Rightarrow 5B+2D=1$

$4B+D=1 \rightarrow$

$$5B+2\cancel{D}=1$$
$$-8B-\cancel{2}D=-2$$
$-3B=-1 \quad B=-1/3$

$4(-1/3)+D=1$

$-4/3+D=1 \Rightarrow D=7/3$

$16A+8B+6C+3D=1$

$16A+8(-1/3)+6C+3(7/3)=1$

$16A+6C-8/3+21/3=1 \Rightarrow 16A+6C+13/3=1 \Rightarrow 16A+6C=-10/3$

$5A+5B+2C+2D=1$

$5A+5(-1/3)+2C+2(7/3)=1 \Rightarrow 5A+2C-5/3+14/3=1$

$5A+2C+9/3=1 \Rightarrow 5A+2C+3=1$

$5A+2C=-2$

$16A+6C=-10/3$

$5A+2C=-2 \Rightarrow$

$16A+6\cancel{C}=-10/3$

$\times -3 \quad -15A-\cancel{6C}=6=18/3$

$A=8/3$

$5(8/3)+2C=-2$

$\frac{40}{3}+2C=-2-\frac{40}{3} \Rightarrow 2C=-\frac{46}{3} \Rightarrow C=-\frac{23}{3}$

$A=8/3,\ B=-1/3,\ C=-\frac{23}{3},\ D=7/3$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4)}\right\}=\frac{8}{3}\cos t-\frac{1}{3}\sin t-\frac{23}{3}\cos 2t+\frac{7}{6}\sin 2t$$

ix. $\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{5s}\right\}=\mathcal{L}^{-1}\left\{e^{-3s}\left(\frac{1}{5s}\right)\right\}=\frac{1}{5}\mathcal{L}^{-1}\left\{e^{-3s}\cdot\frac{1}{s}\right\}$

$a=3$

$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}=1 \Rightarrow \frac{1}{5}(1)\,u(t-3) \Rightarrow \begin{cases}0 & 0\le t\le 3\\ 1/5 & t>3\end{cases}$

$=f(t)$

e i. $\mathcal{L}\{e^{2t}(t-3)^2\}$ translation $a=(-2)$ $= F(s+2)$

$\mathcal{L}\{(t-3)^2\} = \mathcal{L}\{t^2-6t+9\} = \mathcal{L}\{t^2\} - 6\mathcal{L}\{t\} + 9\mathcal{L}\{1\}$

$= \frac{2}{s^3} - 6\cdot\frac{1}{s^2} - 9\cdot\frac{1}{s} = \frac{2}{s^3} - \frac{6}{s^2} - \frac{9}{s} = \frac{2-6s-9s^2}{s^3} = F(s)$

$F(s+2) = \frac{2-6(s+2)-9(s+2)^2}{(s+2)^3}$

ii. $\mathcal{L}\{e^{-4t}\sin(5t)\}$ - translation $a=4$ $= F(s-4)$

$\mathcal{L}\{\sin 5t\} = \frac{5}{s^2+25} = F(s)$

$k=5$

$F(s-4) = \frac{5}{(s-4)^2+25}$

iii. $\mathcal{L}\{(t-1)\,\mathcal{U}(t-1)\} = e^{-s}F(s) = \frac{e^{-s}}{s^2}$

$f(t)=t$ $\quad a=1$

$\mathcal{L}\{t\} = \frac{1}{s^2} = F(s)$

iv. $\mathcal{L}\{(3t+1)\,\mathcal{U}(t-2)\} = e^{-2s}F(s)$

$a=2$

$f(t-2) = 3t+1 = 3(t-2)+1+6 = 3(t-2)+7$

$f(t) = 3t+7$ $\quad \mathcal{L}\{3t+7\} = \frac{3}{s^2}+\frac{7}{s} = F(s)$

$= e^{-2s}\left(\frac{3}{s^2}+\frac{7}{s}\right)$

v. $\mathcal{L}\{\sin(t)\,\mathcal{U}(t-\pi/2)\} = e^{-\pi/2 s}F(s) = \mathcal{L}\{\cos(t-\pi/2)\,\mathcal{U}(t-\pi/2)\}$

$a=\pi/2$ $\quad \mathcal{L}\{\cos t\} = \frac{s}{s^2+1}$

$\sin(t) = \cos(\pi/2 - t) = \cos(t-\pi/2)$ $\quad \Rightarrow \frac{se^{-\pi/2 s}}{s^2+1}$

e. vii. $\mathcal{L}\{f(t)\}$ for $f(t) = \begin{cases} t & 0 \le t \le 1 \\ 1 & t > 1 \end{cases}$

$$f(t) = t - t\,\mathcal{U}(t-1) + 1\,\mathcal{U}(t-1) =$$

$$t - [\underbrace{(t-1)+1}_{T+1}]\,\mathcal{U}(t-1) + \mathcal{U}(t-1) \longrightarrow \text{or use } e^{-as}\mathcal{L}\{g(t+a)\} \text{ gives same result.}$$

$$\mathcal{L}\{f(t)\} = \frac{1}{s^2} - \left(\frac{1}{s^2} + \frac{1}{s}\right)e^{-s} + \frac{e^{-s}}{s}$$

$$= \frac{1}{s^2} - \left(\frac{1+s}{s^2}\right)e^{-s} + \frac{e^{-s}}{s} = \frac{1 - e^{-s} - se^{-s} + se^{-s}}{s^2}$$

$$= \frac{1 - e^{-s}}{s^2}$$

vii $\mathcal{L}\{g(t)\}$ for $g(t) = \begin{cases} \sin(t) & 0 \le t \le \pi \\ \cos t & t > \pi \end{cases}$

$$g(t) = \sin t - \sin t\,\mathcal{U}(t-\pi) + \cos(t-\pi)\,\mathcal{U}(t-\pi)$$

$$\mathcal{L}\{\sin t\} = \frac{1}{s^2+1} \qquad \mathcal{L}\{\sin t\,\mathcal{U}(t-\pi)\} = e^{-\pi s}\mathcal{L}\{\sin(t+\pi)\} = -e^{-\pi s}\frac{1}{s^2+1}$$

$$\mathcal{L}\{\cos(t-\pi)\,\mathcal{U}(t-\pi)\} = e^{-\pi s}\mathcal{L}\{\cos t\} = e^{-\pi s}\frac{s}{s^2+1}$$

$$\mathcal{L}\{g(t)\} = \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1} + \frac{se^{-\pi s}}{s^2+1} = \frac{1 + e^{-\pi s} + se^{-\pi s}}{s^2+1}$$

viii. $\mathcal{L}^{-1}\left\{\frac{1}{s^2-6s+10}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s^2-6s+9)+1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2+1}\right\}$

$$= e^{3t}\sin t$$

ix. $\mathcal{L}^{-1}\left\{\frac{2s+5}{s^2+6s+34}\right\} = \mathcal{L}^{-1}\left\{\frac{2s+5}{(s^2+6s+9)+25}\right\} = \mathcal{L}^{-1}\left\{\frac{2s+5}{(s+3)^2+25}\right\} =$

$$\mathcal{L}^{-1}\left\{\frac{2(s+3)-6+5}{(s+3)^2+25}\right\} = 2\mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2+25}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2+25}\right\} =$$

eix cont'd

$$2\mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2+25}\right\} - \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{5}{(s+3)^2+25}\right\} =$$

$$2e^{-3t}\cos 5t - \frac{1}{5}e^{-3t}\sin 5t$$

x. $$\mathcal{L}^{-1}\left\{\frac{(s+1)^2}{(s+2)^4}\right\} = \mathcal{L}^{-1}\left\{\frac{s^2+2s+1}{(s+2)^4}\right\} = \mathcal{L}^{-1}\left\{\frac{A}{(s+2)} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)^3} + \frac{D}{(s+2)^4}\right\}$$

$$A(s+2)^3 + B(s+2)^2 + C(s+2) + D = (s+1)^2$$

$s=-2$ $\quad D = (-2+1)^2 = 1$

$s=0$ $\quad 8A + 4B + 2B = 1 - 1 = 0$

$s=-1$ $\quad A + B + C + 1 = 0$

$s=-3$ $\quad -A + B - C + 1 = 4$

$\quad B + 2 = 4 \Rightarrow B = 2$

$$4A + 2B + \cancel{C} = 0$$
$$-A \mp B \mp \cancel{C} = +1$$
$$3A + B = 1$$
$$3A + 2 = 1 \Rightarrow 3A = -1 \Rightarrow A = -1/3$$

$-1/3 + 2 + C = -1$

$-1/3 + C = -3 \Rightarrow C = -3 + 1/3 = -8/3$

$$-\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\} - 4/3\,\mathcal{L}^{-1}\left\{\frac{2}{(s+2)^3}\right\} + \frac{1}{6}\mathcal{L}^{-1}\left\{\frac{6}{(s+2)^4}\right\}$$

$$= -\frac{1}{3}e^{-2t} + 2te^{-2t} - 4/3\,t^2e^{-2t} + \frac{1}{6}t^3e^{-2t}$$

xi. $$\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\} = \mathcal{L}^{-1}\left\{\underset{\substack{\uparrow \\ a}}{e^{-\pi s}} \cdot \frac{1}{s^2+1}\right\} = \sin(t-\pi)\,\mathcal{U}(t-\pi)$$

xii. $$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2(s-1)}\right\} = \mathcal{L}^{-1}\left\{e^{-2s} \cdot \frac{1}{s^2(s-1)}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s-1)}\right\} = \mathcal{L}^{-1}\left\{\frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1}\right\} = -2 + t + \frac{1}{2}t^2$$

$$As(s-1) + B(s-1) + Cs^2 = 1$$

$s=0$ $\quad -1B = 1 \Rightarrow B = 1$

$s=1$ $\quad C = 1$

$s=2$ $\quad 2A + B + 4C = 1 \Rightarrow 2A + \cancel{1} + 4 = \cancel{1} \Rightarrow 2A = -4 \Rightarrow A = -2$

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2(s-1)}\right\} = \left[-2 + (t-2) + \frac{1}{2}(t-2)^2\right]\mathcal{U}(t-2)$$

xiii. $\mathcal{L}^{-1}\left\{\frac{se^{-s}}{s^2+49}\right\} = \mathcal{L}^{-1}\left\{e^{-s}\cdot\frac{s}{s^2+49}\right\} =$

$a=+1$

$\mathcal{L}^{-1}\left\{\frac{s}{s^2+49}\right\} = \cos 7t \Rightarrow \cos[7(t-1)]\mathcal{U}(t-1)$

fi. $\mathcal{L}\{te^{-3t}\cos 4t\} = (-1)\frac{d}{ds}\left[\mathcal{L}\{e^{-3t}\cos 4t\}\right] = (-1)\frac{d}{ds}\left[\frac{s+3}{(s+3)^2+16}\right]$

$=(-1)\frac{(1)[(s+3)^2+16]-2(s+3)(s+3)}{[(s+3)^2+16]^2} = (-1)\frac{(s+3)^2+16-2(s+3)^2}{[(s+3)^2+16]^2} = (-1)\frac{-(s+3)^2+16}{[(s+3)^2+16]^2} =$

$\frac{(s+3)^2-16}{[(s+3)^2+16]^2}$

ii. $\mathcal{L}\{t\sinh t\} = (-1)\frac{d}{ds}[\mathcal{L}\{\sinh t\}] = (-1)\frac{d}{ds}\left[\frac{1}{s^2-1}\right] =$

$(-1)\frac{d}{ds}(s^2-1)^{-1} = (-1)(-1)(s^2-1)^{-2}(2s) = \frac{2s}{(s^2-1)^2}$

iii. $\mathcal{L}\{t^2\sin t\} = (-1)^2\frac{d^2}{ds^2}[\mathcal{L}\{\sin t\}] = \frac{d^2}{ds^2}\left[\frac{1}{s^2+1}\right] = \frac{d^2}{ds^2}(s^2+1)^{-1}$

$\frac{d}{ds}(-1)(s^2+1)^{-2}2s = (-1)(-2)(s^2+1)^{-3}\cdot 2s\cdot 2s + (-1)(s^2+1)^{-2}(2) =$

$\frac{8s}{(s^2+1)^3} - \frac{2}{(s^2+1)^2}$

iv. $\mathcal{L}\{t^2 * \cos 2t\} = \mathcal{L}\{t^2\}\cdot\mathcal{L}\{\cos 2t\} = \frac{2}{s^3}\cdot\frac{s}{s^2+4}$

v. $\mathcal{L}\{e^{5t} * \cosh t\} = \mathcal{L}\{e^{5t}\}\cdot\mathcal{L}\{\cosh t\} = \frac{1}{s-5}\cdot\frac{s}{s^2-1}$

vi. $\mathcal{L}\{t * e^{4t}\} = \mathcal{L}\{t\}\cdot\mathcal{L}\{e^{4t}\} = \frac{1}{s^2}\cdot\frac{1}{s-4}$

vii. $\mathcal{L}\left\{\int_0^t \sin\tau\, d\tau\right\}$ $\quad f(\tau) = \sin\tau \quad g(t-\tau) = 1$

$=\mathcal{L}\{\sin t * 1\} = \mathcal{L}\{\sin t\}\cdot\mathcal{L}\{1\} = \frac{1}{s^2+1}\cdot\frac{1}{s}$

viii. $\mathcal{L}\left\{t\int_0^t \cosh\tau\, d\tau\right\} = (-1)\frac{d}{ds}\left[\mathcal{L}\left\{\int_0^t \cosh\tau\, d\tau\right\}\right]$

$f(\tau) = \cosh\tau \quad g(t-\tau) = 1$

$= (-1)\frac{d}{ds}\left[\frac{\cancel{s}}{s^2-1}\cdot\frac{1}{\cancel{s}}\right] = (-1)\frac{d}{ds}(s^2-1)^{-1} = (-1)(-1)(s^2-1)^{-2}\cdot 2s =$

$\frac{2s}{(s^2-1)^2}$

f. ix. $\mathcal{L}\{\int_0^t \tau^3 \sin(t-\tau)\,d\tau\}$ $\quad f(\tau)=\tau^3 \quad g(t-\tau)=\sin(t-\tau)$ 13

$= \mathcal{L}\{t^3\}\cdot\mathcal{L}\{\sin t\} = \frac{3!}{s^4}\cdot\frac{1}{s^2+1} = \frac{6}{s^4(s^2+1)}$

x. $\mathcal{L}^{-1}\{\frac{1}{s(s+1)}\} = \mathcal{L}^{-1}\{\frac{1}{s}\cdot\frac{1}{s+1}\} = \int_0^t e^{-\tau}\,d\tau$

$\mathcal{L}^{-1}\{\frac{1}{s}\} = 1 = g(t-\tau) \quad \mathcal{L}^{-1}\{\frac{1}{s+1}\} = e^{-\tau} = f(\tau)$

xi. $\mathcal{L}^{-1}\{\frac{1}{s^2(s-1)}\} = \mathcal{L}^{-1}\{\frac{1}{s^2}\cdot\frac{1}{s-1}\} = \int_0^t \tau e^{t-\tau}\,d\tau$

$\mathcal{L}^{-1}\{\frac{1}{s^2}\} = t = f(t) \quad \mathcal{L}^{-1}\{\frac{1}{s-1}\} = e^t = g(t)$

xii. $\mathcal{L}^{-1}\{\frac{2}{(s+3)^2}\} = 2te^{-3t}$

$\int 2(s+3)^{-2}\,ds \Rightarrow \frac{2(s+3)^{-1}}{-1}(-1) = \frac{2}{s+3}$

$\mathcal{L}^{-1}\{\frac{2}{s+3}\} = 2\mathcal{L}^{-1}\{\frac{1}{s+3}\} = 2e^{-3t}$ $\quad \frac{d}{ds}$ accounts for t

xiii. $\mathcal{L}^{-1}\{\frac{s}{s^2(s^2+4)}\} = \mathcal{L}^{-1}\{\frac{1}{s^2}\cdot\frac{s}{s^2+4}\} = \int_0^t \tau\cos[2(t-\tau)]\,d\tau$

$\mathcal{L}^{-1}\{\frac{1}{s^2}\} = t = f(t) \quad \mathcal{L}^{-1}\{\frac{s}{s^2+4}\} = \cos 2t = g(t)$

g. xiv. $\mathcal{L}\{f(t)\}$ for $f(t) = \begin{cases} -1 & 0 \le t \le 1 \\ 1 & t > 1 \end{cases}$

$f(t) = -1 - (-1)\,\mathcal{U}(t-1) + 1\,\mathcal{U}(t-1) = -1 + 2\mathcal{U}(t-1)$

$\mathcal{L}\{f(t)\} = -\frac{1}{s} + \frac{2e^{-s}}{s}$

xv. $f(t) = \begin{cases} 2t+1 & 0 \le t < 1 \\ 0 & t \ge 1 \end{cases}$

$f(t) = 2t-1 - (2t-1)\,\mathcal{U}(t-1) + 0\,\mathcal{U}(t-1)$

$[2(t-1)+1]\,\mathcal{U}(t-1)$

$\mathcal{L}\{f(t)\} = \frac{2}{s^2} - \frac{1}{s} - [e^{-s}(\frac{2}{s^2} + \frac{1}{s})]$

g. xvi. $\mathcal{L}\{te^{4t}\} = \frac{1}{(s-4)^2}$

xvii. $\mathcal{L}\{t^5\} = \frac{120}{s^6}$

xviii. $\mathcal{L}\{\cos 5t + \sin 2t\} = \frac{s}{s^2+25} + \frac{2}{s^2+4}$

xix. $\mathcal{L}\{\cosh 4t\} = \frac{s}{s^2-16}$

xx. $\mathcal{L}\{e^t \sin 3t\} = \frac{3}{(s-1)^2+9}$

xxi. $\mathcal{L}\{e^{2t}(t-1)^2\} = \mathcal{L}\{e^{2t}[t^2-2t+1]\} =$

$$\frac{2}{(s-2)^3} - \frac{2}{(s-2)^2} + \frac{1}{(s-2)}$$

xxii. $\mathcal{L}\{t^2 * te^t\} = \mathcal{L}\{t^2\} \cdot \mathcal{L}\{te^t\} = \frac{2}{s^3} \cdot \frac{1}{(s-1)^2}$

xxiii. $\mathcal{L}\{\int_0^t \tau \cos\tau \, d\tau\}$ $\quad f(\tau) = \tau\cos\tau \quad g(t-\tau) = 1$

$$= (-1)\frac{d}{ds}\left[\frac{s}{s^2+1}\right] \cdot \frac{1}{s} = (-1)\left[\frac{1(s^2+1) - 2s(s)}{(s^2+1)^2}\right]\frac{1}{s} =$$

$$\frac{s^2-1}{(s^2+1)^2} \cdot \frac{1}{s}$$

xxN. $\mathcal{L}\{\int_0^t \tau e^{t-\tau}\}$ $\quad f(\tau) = \tau \quad g(t-\tau) = e^{t-\tau}$

$g(t) = e^t$

$$= \mathcal{L}\{t\} \cdot \mathcal{L}\{e^t\} = \frac{1}{s^2} \cdot \frac{1}{s-1}$$

xxv. $\mathcal{L}^{-1}\{\frac{1}{s^2}\} = t$

xxvi. $\mathcal{L}^{-1}\{\frac{1}{4s+1}\} = \frac{1}{4}\mathcal{L}^{-1}\{\frac{1}{s+\frac{1}{4}}\} = \frac{1}{4}e^{-1/4 t}$

xxvii. $\mathcal{L}^{-1}\{\frac{5}{s^2+49}\} = \frac{5}{7}\mathcal{L}^{-1}\{\frac{7}{s^2+49}\} = \frac{5}{7}\sin 7t$

xxviii. $\mathcal{L}^{-1}\left\{\frac{2s-6}{s^2+9}\right\} = 2\left[\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} - \mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}\right] =$ 15

$2\cos 3t - 2\sin 3t$

xxix. $\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s-3}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{(s+3)(s-1)}\right\} = \mathcal{L}^{-1}\left\{\frac{A}{s+3} + \frac{B}{s-1}\right\}$

$A(s-1) + B(s+3) = s$

$s=1$ $\quad 4B = 1 \Rightarrow B = 1/4$ $\qquad s=-3 \quad -2A = -3 \Rightarrow A = \frac{3}{2}$

$= \frac{3}{2}e^{-3t} + \frac{1}{4}e^{t}$

xxx. $\mathcal{L}^{-1}\left\{\frac{2s-4}{(s^2+s)(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{A}{s} + \frac{B}{s+1} + \frac{Cs+D}{s^2+1}\right\}$

$A(s+1)(s^2+1) + B(s)(s^2+1) + (Cs+D)(s)(s+1) = 2s-4$

$s=0 \quad A = -4$

$s=-1 \quad -2B = -6 \Rightarrow B = 3$

$s=1 \quad 4A + 2B + (C+D)(2) = -2$

$4(-4) + 2(3) + 2C + 2D = -2$

$-16 + 6 + 2C + 2D = -2$

$-10 + 2C + 2D = -2$

$+10 \qquad +10$

$2C + 2D = 8 \Rightarrow C + D = 8$

$s=2 \quad A(3)(5) + B(2)(5) + (C(2) + D)(2)(3) = 0$

$15(-4) + 10(3) + 12C + 6D = 0$

$-60 + 30 + 12C + 6D = 0$

$-30 + 12C + 6D = 0$

$12C + 6D = 30 \Rightarrow$

$2C + \cancel{D} = 5$

$-C - \cancel{D} = -8$

$C = -3$

$-3 + D = 8$

$D = 11$

$\mathcal{L}^{-1}\left\{-\frac{4}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{3}{s+1}\right\} + \mathcal{L}^{-1}\left\{-\frac{3s}{s^2+1}\right\}$

$+ \mathcal{L}^{-1}\left\{\frac{11}{s^2+1}\right\} = -4\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - 3\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}$

$+ 11\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = -4 + 3e^{-t} - 3\cos t + 11\sin t$

g.

xxxi. $\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s^2+2s+1)+4}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+4}\right\}$

$e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}e^{-t}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \frac{1}{2}e^{-t}\sin 2t.$

xxxii. $\mathcal{L}^{-1}\left\{\frac{(s+1)^2}{(s+2)^4}\right\}$ see e.x.

$= -\frac{1}{3}e^{-2t} + 2te^{-2t} - \frac{1}{3}t^2e^{-2t} - \frac{1}{6}t^3e^{-2t}$

xxxiii. $\mathcal{L}^{-1}\left\{\frac{1}{s(s-1)}\right\} = \mathcal{L}^{-1}\left\{\frac{A}{s} + \frac{B}{s-1}\right\} = 1 + e^t$

$A(s-1) + Bs = 1$

$s=0 \quad -A = 1 \Rightarrow A = -1 \qquad s=1 \quad B=1$

h. i. $y' - y = 1,\ y(0) = 0 \qquad \mathcal{L}\{y(t)\} = Y(s)$

$sY(s) - 0 - Y(s) = \frac{1}{s} \Rightarrow (s-1)Y(s) = \frac{1}{s}$

$\Rightarrow Y(s) = \frac{1}{s(s-1)} \qquad \mathcal{L}^{-1}\left\{\frac{1}{s(s-1)}\right\} = y(t) = 1 + e^t$

(see g. xxxiii.)

ii. $y' + 6y = e^{4t},\ y(0) = 2$

$sY(s) - \cancel{2} + 6Y(s) = \frac{1}{s-4} + 2$

$Y(s)(s+6) = \frac{1}{s-4} + \frac{2(s-4)}{s-4} \Rightarrow Y(s) = \frac{2s-7}{(s-4)(s+6)}$

$\frac{2s-7}{(s-4)(s+6)} = \frac{A}{s-4} + \frac{B}{s+6}$

$A(s+6) + B(s-4) = 2s - 7$

$s=-6 \quad -10B = -19 \qquad s=4 \quad 10A = 1$

$B = \frac{19}{10} \qquad A = \frac{1}{10}$

$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{10}\cdot\frac{1}{s-4} + \frac{19}{10}\cdot\frac{1}{s+6}\right\} = \frac{1}{10}e^{4t} + \frac{19}{10}e^{-6t}$

iii. $y'' + 5y' + 4y = 0 \quad y(0) = 1,\ y'(0) = 0$

$s^2Y(s) - s - 0 + 5sY(s) - 5 + 4Y(s) = 0$

ii iii cont'd

$$Y(s)(s^2+5s+4) = s+5$$

$$Y(s) = \frac{s+5}{s^2+5s+4} = \frac{s+5}{(s+1)(s+4)} = \frac{A}{s+1} + \frac{B}{s+4}$$

$$A(s+4) + B(s+1) = s+5$$

$s=-1 \quad 3A = 4 \Rightarrow A = \frac{4}{3} \qquad s=-4 \quad -3B = 1 \Rightarrow B = -\frac{1}{3}$

$$y(t) = \tfrac{4}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} = \frac{4}{3}e^{-t} - \frac{1}{3}e^{-4t}$$

iv. $y'' - 4y' = 6e^{3t} - 3e^{-t}, \; y(0)=1, \; y'(0) = -1$

$$s^2Y(s) - s + 1 - 4sY(s) + 4 = \frac{6}{s-3} - \frac{3}{s+1} = \frac{6s+6-3s+9}{(s-3)(s+1)} = \frac{3s+15}{(s-3)(s+1)}$$

$$Y(s)(s^2-4s) = \frac{3s+15}{(s-3)(s+1)} + s - 4 \Rightarrow$$

$$Y(s) = \frac{3s+15}{s(s-4)(s-3)(s+1)} + \frac{\cancel{s-4}}{s\cancel{(s-4)}}$$

$$\frac{A}{s} + \frac{B}{s-4} + \frac{C}{s-3} + \frac{D}{s+1} + \frac{1}{s}$$

$$A(s-4)(s-3)(s+1) + Bs(s-3)(s+1) + Cs(s-4)(s+1) + Ds(s-4)(s-3) = 3s+15$$

$s=0 \quad A(-4)(-3)(1) = 15 \Rightarrow A = \frac{15}{12} = \frac{5}{4} + 1 = \frac{9}{4}$ (for $\frac{1}{s}$)

$s=4 \quad B(4)(-1)(5) = 27 \Rightarrow -20B = 27 \Rightarrow B = \frac{-27}{20}$

$s=3 \quad C(3)(-1)(4) = 24 \Rightarrow -12C = 24 \Rightarrow C = -2$

$s=-1 \quad D(-1)(-5)(-4) = 12 \Rightarrow -20D = 12 \Rightarrow D = -\frac{12}{20} = -\frac{3}{5}$

$$y(t) = \tfrac{9}{4}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{27}{20}\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} - \frac{3}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= \frac{9}{4} - \frac{27}{20}e^{4t} - 2e^{3t} - \frac{3}{5}e^{-t}$$

v. $y'' - 4y' + 4y = t^3e^{2t}, \; y(0) = 0, \; y'(0) = 0$

$$s^2Y(s) - 0 - 0 - 4sY(s) - 0 + 4Y(s) = \frac{6}{(s-2)^4}$$

$$Y(s)\underbrace{(s^2-4s+4)}_{(s-2)^2} = \frac{6}{(s-2)^4} \Rightarrow$$

$$Y(s) = \frac{6}{(s-2)^6} \qquad y(t) = \frac{1}{20}\mathcal{L}^{-1}\left\{\frac{5!}{(s-2)^6}\right\} = \frac{1}{20}t^5e^{2t}$$

h. vi. $y'' - y' = e^t \cos t$, $y(0)=0$, $y'(0)=0$

$$s^2Y(s) - 0 - 0 - sY(s) + 0 = \frac{s-1}{(s-1)^2+1} = \frac{s-1}{s^2-2s+1+1} = \frac{s-1}{s^2-2s+2}$$

$$Y(s)(s^2-s) = \frac{\cancel{s-1}}{s(\cancel{s-1})[(s-1)^2+1]} = \frac{1}{s[(s-1)^2+1]} = \frac{A}{s} + \frac{Bs+C}{[(s-1)^2+1]}$$

$$A(s^2-2s+2) + (Bs+C)(s) = 1$$

$s=0$ $\quad 2A = 1 \Rightarrow A = 1/2$ $\qquad s=1 \quad A(1) + B + C = 1 \Rightarrow$

$B + C = 1/2$

$s=-1$ $\quad A(1+2+2)+(-B+C)(-1) = 1 \Rightarrow 5A + B - C = 1 \Rightarrow \frac{5}{2} + B - C = 1 \Rightarrow B - C = -\frac{3}{2}$

$B + C = 1/2$

$B + C = 1/2 \Rightarrow -1/2 + C = 1/2 \Rightarrow C = 1$ $\qquad 2B = -1$

$B = -1/2$

$$y(t) = 1/2\, \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{-1/2\, s + 1}{s^2-2s+2}\right\} =$$

$$\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s-2}{(s-1)^2+1}\right\} = \qquad s-2 = s-1-1$$

$$\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{2}\left[\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+1}\right\}\right]$$

$$= \frac{1}{2} - \frac{1}{2}e^t \cos t + \frac{1}{2}e^t \sin t.$$

vii. $y' + 2y = f(t)$, $y(0)=0$, $f(t) = \begin{cases} t & 0 \le t < 1 \\ 0 & t \ge 1 \end{cases} = t - t\,\mathcal{U}(t-1) + 0$

$g(t+1) = t+1$

$$sY(s) - 0 + 2Y(s) = \frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right)$$

$$Y(s)(s+2) = \frac{1}{s^2} - \frac{e^{-s}(1+s)}{s^2} = \frac{1 - e^{-s}(1+s)}{s^2}.$$

$$Y(s) = \frac{1 - e^{-s}(s+1)}{s^2(s+2)} = \frac{1}{s^2(s+2)} + e^{-s}\left(\frac{s+1}{s^2(s+2)}\right)$$

$$\frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2} = \frac{1}{s^2(s+2)}$$

$$As(s+2) + B(s+2) + Cs^2 = 1$$

$s=0$ $\quad 2B = 1 \Rightarrow B = 1/2$ $\qquad s=-2 \quad 4C = 1 \Rightarrow C = 1/4$

$s=1$ $\quad 3A + 3B + C = 1 \Rightarrow 3A + 3/2 + 1/4 = 1 \Rightarrow 3A = -3/4 \Rightarrow A = -1/4.$

$$\frac{D}{s} + \frac{E}{s^2} + \frac{F}{s+2} = \frac{s+1}{s^2(s+2)}$$

$$Ds(s+2) + E(s+2) + F(s^2) = s+1$$

$s=-2$ $\quad 4F = -1 \Rightarrow F = -1/4$

$s=0$ $\quad 2E = 1 \Rightarrow E = 1/2$

$s=1$ $\quad 3D + 1/2(3) + 1/4 = 2 \Rightarrow 3D = 3/4 \Rightarrow D = 1/4$

h vii cont'd

$$y(t) = -\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \mathcal{L}^{-1}\left\{e^{-s}\left(\frac{1/4}{s} + \frac{1/2}{s^2} - \frac{1/4}{s+2}\right)\right\}$$

$$-\frac{1}{4} + \frac{1}{2}t + \frac{1}{4}e^{-2t} + \left[\frac{1}{4} + \frac{1}{2}(t-1) - \frac{1}{4}e^{-2(t-1)}\right]\mathcal{U}(t-1)$$

$$\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = \frac{1}{4}, \quad \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = \frac{1}{2}t, \quad -\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = -\frac{1}{4}e^{-2t}$$

viii $y'' + 4y = \sin t\, \mathcal{U}(t-2\pi)$, $y(0) = 1$, $y'(0) = 0$ $\quad \sin(t-2\pi) = \sin(t)$

$$s^2 Y(s) - s - 0 + 4Y(s) = e^{-\pi s}\cdot\frac{1}{s^2+1}$$

$$Y(s)(s^2+4) = e^{-\pi s}\cdot\frac{1}{s^2+1} + s$$

$$Y(s) = e^{-\pi s}\cdot\frac{1}{(s^2+1)(s^2+4)} + \frac{s}{s^2+4}$$

see d. viii for decomp.

$$y(t) = \mathcal{L}^{-1}\left\{e^{-\pi s}\left(\frac{As}{s^2+1} + \frac{B}{s^2+1} + \frac{Cs}{s^2+4} + \frac{D}{s^2+4}\right)\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}$$

$$\cos 2t + \left[\frac{8}{3}\cos(t-\pi) - \frac{1}{3}\sin(t-\pi) - \frac{23}{3}\cos 2(t-\pi) + \frac{7}{6}\sin 2(t-\pi)\right]\mathcal{U}(t-\pi)$$

ix. $y'(t) = 1 - \sin(t) - \int_0^t y(\tau)\,d\tau \quad y(0) = 0$ $\quad f(\tau) = y(\tau)$, $g(t-\tau) = 1$

$$sY(s) = \frac{1}{s} - \frac{1}{s^2+1} - Y(s)\cdot\frac{1}{s}$$

$$\left(s + \frac{1}{s}\right)Y(s) = \frac{s^2 - s + 1}{s(s^2+1)} = \left(\frac{s^2+1}{s}\right)Y(s)$$

$$Y(s) = \frac{s^2 - s + 1}{\cancel{s}(s^2+1)}\cdot\frac{\cancel{s}}{s^2+1} = \frac{s^2 - s + 1}{(s^2+1)^2} = \frac{\cancel{s^2+1}}{(s^2+1)^{\cancel{2}}} - \frac{s}{(s^2+1)^2}$$

$$= \frac{1}{s^2+1} - \frac{s}{(s^2+1)^2}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \sin t + \frac{1}{2}t\sin t$$

$$\int \frac{s}{(s^2+1)^2}\,ds \quad u = s^2+1, \ \frac{1}{2}du = \cancel{2}s\,ds \quad \frac{1}{2}\int \frac{1}{u^2}\,du = -\frac{1}{2}\frac{1}{u}\,du \Rightarrow (-1)\left(-\frac{1}{2}\right)\frac{1}{s^2+1}$$