Math 2255 Homework #6 Key



1. $f(t) = \begin{cases} t^2 & 0 \le t \le 1 \\ (t-1)^{-1} & 1 < t \le 2 \\ 1 & 2 < t \le 3 \end{cases}$

graph is piecewise continuous each piece is continuous on the interval it's defined on, but at $t=1$ $(1)^2 = 1$, but $(1-1)^{-1}$ is undefined so there is a jump in the graph here.

2. a. $\mathcal{L}\{t\} = \int_0^\infty e^{-st}\, t\, dt$

| +/- | u | dv |
|---|---|---|
| + | $t$ | $e^{-st}$ |
| − | 1 | $-\frac{1}{s}e^{-st}$ |
| + | 0 | $\frac{1}{s^2}e^{-st}$ |

$= -\frac{1}{s}te^{-st} - \frac{1}{s^2}e^{-st}\Big|_0^\infty =$

$0 - 0 + 0 + \frac{1}{s^2} = \frac{1}{s^2}$

b. $\mathcal{L}\{\cos at\} = \int_0^\infty e^{-st}\cos at\, dt$

$u = \cos at$ $\quad dv = e^{-st}$
$du = -a\sin at$ $\quad v = -\frac{1}{s}e^{-st}$

$-\frac{1}{s}e^{-st}\cos at - \int +\frac{a}{s}e^{-st}\sin at\, dt$

$u = \sin at$ $\quad dv = e^{-st}$
$du = a\cos at$ $\quad v = -\frac{1}{s}e^{-st}$

$-\frac{1}{s}e^{-st}\cos at - \frac{a}{s}\left[-\frac{1}{s}e^{-st}\sin at - \int -\frac{a}{s}e^{-st}\cos at\, dt\right]$

$-\frac{1}{s}e^{-st}\cos at + \frac{a}{s^2}e^{-st}\sin at - \cancel{\frac{a^2}{s^2}\int e^{-st}\cos at\, dt} = \int e^{-st}\cos at\, dt$

$\cancel{+\frac{a^2}{s^2}\int e^{-st}\cos at\, dt}$ $\quad \frac{a^2}{s^2}\int e^{-st}\cos at\, dt$

$\left(\frac{s^2+a^2}{\cancel{s^2}}\right)\int e^{-st}\cos at\, dt$

$\left(-\frac{1}{s}e^{-st}\cos at + \frac{a}{s^2}e^{-st}\sin at\right)\Big|_0^\infty \frac{s^2}{s^2+a^2} =$

$(0 + 0 + \frac{1}{s} - 0)\left(\frac{s^2}{s^2+a^2}\right) = \frac{s}{s^2+a^2}$

2c. $\mathcal{L}\{\sinh bt\} = \int_0^\infty e^{-st}\left(\frac{1}{2}\right)\left(e^{bt}-e^{-bt}\right) dt = \frac{1}{2}\int_0^\infty e^{-(s-b)t} - e^{-(s+b)t}\, dt$

$$= \frac{1}{2}\left(\frac{-1}{s-b}e^{-(s-b)t} + \frac{1}{s+b}e^{-(s+b)t}\right)\Big|_0^\infty = \frac{1}{2}\left(0+0+\frac{1}{s-b}-\frac{1}{s+b}\right) =$$

$$\frac{1}{2}\left(\frac{s+b-(s-b)}{(s-b)(s+b)}\right) = \frac{1}{\cancel{2}}\left(\frac{\cancel{2}b}{s^2-b^2}\right) = \frac{b}{s^2-b^2}$$

d. $\mathcal{L}\{e^{at}\cos bt\} = \int_0^\infty e^{-st}e^{at}\cos bt\, dt = \int_0^\infty e^{-(s-a)t}\cos bt\, dt$

by analogy w/ (b) $\Rightarrow$ $\frac{s-a}{(s-a)^2+a^2}$

e. $\mathcal{L}\{te^{at}\} = \int_0^\infty e^{-st}\, te^{at}\, dt = \int_0^\infty te^{-(s-a)t}\, dt =$

by analogy w/ (a) $\Rightarrow$ $\frac{1}{(s-a)^2}$

f. $\mathcal{L}\{t^2\sinh bt\} = \int_0^\infty t^2e^{-st}\left(\frac{1}{2}\right)\left(e^{bt}-e^{-bt}\right) dt =$

$$\frac{1}{2}\int_0^\infty t^2e^{-(s-b)t} - t^2e^{-(s+b)t}\, dt$$

| $\pm$ | $u$ | $dv$ |
|---|---|---|
| $+$ | $t^2$ | $e^{-kt}$ |
| $-$ | $2t$ | $-\frac{1}{k}e^{-kt}$ |
| $+$ | $2$ | $\frac{1}{k^2}e^{-kt}$ |
| $-$ | $0$ | $-\frac{1}{k^3}e^{-kt}$ |

$$\frac{1}{2}\Big[\frac{-t^2}{s-b}e^{-(s-b)t} - \frac{2t}{(s-b)^2}e^{-(s-b)t} - \frac{2}{(s-b)^3}e^{-(s-b)t} + \frac{t^2}{(s+b)}e^{-(s+b)t} + \frac{2t}{(s+b)^2}e^{-(s+b)t} + \frac{2}{(s+b)^3}e^{-(s+b)t}\Big]_0^\infty$$

$$\frac{1}{2}\left[0-0-0+0+0+0+0+0+\frac{2}{(s-b)^3}-0-0-\frac{2}{(s+b)^3}\right] =$$

$$\frac{1}{(s-b)^3}-\frac{1}{(s+b)^3} = \frac{(s+b)^3-(s-b)^3}{(s^2-b^2)^3} = \frac{\cancel{s^3}+3s^2b+\cancel{3sb^2}+b^3-\cancel{s^3}+3s^2b-\cancel{3sb^2}+b^3}{(s^2-b^2)^3}$$

$$= \frac{6s^2b+2b^3}{(s^2-b^2)^3} = \frac{2b(3s^2+b^2)}{(s^2-b^2)^3}$$

2g. $\mathcal{L}\{u_c(t)\} = \int_0^\infty u_c(t)e^{-st}\,dt = \int_0^c 0\cdot e^{-st}\,dt + \int_c^\infty 1\,e^{-st}\,dt$ (3)

$= -\frac{1}{s}e^{-st}\Big|_c^\infty = 0 + \frac{1}{s}e^{-sc}$

3. a. $\mathcal{L}^{-1}\left\{\frac{3}{s^2+4}\right\} = \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \frac{3}{2}\sin 2t$

b. $\mathcal{L}^{-1}\left\{\frac{3s}{s^2-s-6}\right\} = \mathcal{L}^{-1}\left\{\frac{A}{s-3} + \frac{B}{s+2}\right\} = \frac{9}{5}e^{3t} + \frac{6}{5}e^{-2t}$

$(s-3)(s+2)$

$A(s+2) + B(s-3) = 3s$

$s=-2 \quad -5B=-6 \Rightarrow B = 6/5 \qquad s=3 \quad 5A = 9 \Rightarrow A = 9/5$

c. $\mathcal{L}^{-1}\left\{\frac{8s^2-4s+12}{s(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{A}{s} + \frac{Bs+C}{s^2+4}\right\}$

$A(s^2+4) + (Bs+C)(s) = 8s^2-4s+12$

$s=0 \quad 4A = 12 \Rightarrow A=3 \qquad s=1 \quad 15 + B + C = 8-4+12 = 16$

$B + C = 1$

$s=-1 \quad 15 + (-B+C)(-1) = 8+4+12 \Rightarrow \quad B - C = 24-15 = 9$

$2B = 10 \Rightarrow B=5 \Rightarrow C=-4$

$3\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 5\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - 2\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} =$

$3 + 5\cos 2t - 2\sin 2t$

d. $\mathcal{L}^{-1}\left\{\frac{2s-3}{s^2+2s+10}\right\} = \mathcal{L}^{-1}\left\{\frac{2s-3}{(s^2+2s+1)+9}\right\} = \mathcal{L}^{-1}\left\{\frac{2(s+1)-5}{(s+1)^2+9}\right\} =$

$2\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+9}\right\} - \frac{5}{3}\mathcal{L}^{-1}\left\{\frac{3}{(s+1)^2+9}\right\} = 2e^{-t}\cos 3t - \frac{5}{3}e^{-t}\sin 3t$

4a. $y'' - y' - 6y = 0 \quad y(0)=1,\ y'(0)=-1$

$s^2Y(s) - s(1) + 1 - sY(s) + 1 - 6Y(s) = 0$

$Y(s)(s^2-s-6) = s-1 \Rightarrow Y(s) = \frac{s-1}{(s-3)(s+2)}$

$(s-3)(s+2)$

4a cont'd

$$\mathcal{L}^{-1}\left\{\frac{s-1}{(s-3)(s+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{A}{s-3} + \frac{B}{s+2}\right\} = \frac{2}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + \frac{3}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$A(s+2) + B(s-3) = s-1$$

$s=-2 \quad -5B = -3 \Rightarrow B = \frac{3}{5} \qquad s=3 \quad 5A = 2 \Rightarrow A = 2/5$

$$y(t) = \frac{2}{5}e^{3t} + \frac{3}{5}e^{-2t}$$

b. $y^{IV} - y = 0, \quad y(0)=1, \; y'(0)=0, \; y''(0)=1, \; y'''(0)=0$

$$s^4 Y(s) - s^3(1) - s^2(0) - s(1) - 0 - Y(s) = 0$$

$$Y(s)(\cancel{s^4-1}) = \frac{s^3+s}{s^4-1} = \frac{s\cancel{(s^2+1)}}{\cancel{(s^2+1)}(s^2-1)} = \frac{s}{s^2-1}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2-1}\right\} = \cosh t = y(t)$$

c. $y'' + 2y' + y = 4e^{-t}, \quad y(0)=2, \; y'(0)=-1$

$$s^2Y(s) - 2s + 1 + 2sY(s) - 2 + Y(s) = \frac{4}{s-1}$$

$$Y(s)\underbrace{(s^2+2s+1)}_{(s+1)^2} = \frac{4}{s-1} + \frac{(2s+1)(s-1)}{s-1} = \frac{4+2s^2-2s+s-1}{s-1} = \frac{2s^2-s+3}{s-1}$$

$$Y(s) = \frac{(2s-3)\cancel{(s+1)}}{(s+1)^{\cancel{2}}(s-1)} = \frac{2s-3}{(s+1)(s-1)} = 2\left(\frac{s}{s^2-1}\right) - 3\left(\frac{1}{s^2-1}\right)$$

$$y(t) = 2\mathcal{L}^{-1}\left\{\frac{s}{s^2-1}\right\} - 3\mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} = 2\cosh t - 3\sinh t$$

d. $y'' + 4y = \begin{cases} 1 & 0 \le t \le \pi \\ 0 & t > \pi \end{cases}, \quad y(0)=1, \; y'(0)=0$

$$= 1 + \begin{cases} 0 & 0 \le t < \pi \\ -1 & t > \pi \end{cases} = 1 - u(t-\pi)$$

$$s^2Y(s) - s - 0 + 4Y(s) = \frac{1}{s} - \frac{e^{-\pi s}}{s}$$

$$Y(s)(s^2+4) = \left(\frac{1}{s} + s\right) - \frac{e^{-\pi s}}{s} = \frac{1+s}{s} - \frac{e^{-\pi s}}{s}$$

4d. cont'd

⑤

$$Y(s) = \frac{1+s}{s(s^2+4)} - e^{-\pi s}\left(\frac{1}{s(s^2+4)}\right)$$

$$\frac{1+s}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$$

$$A(s^2+4) + (Bs+C)(s) = 1+s$$

$s=0$ $\quad 4A = 1 \Rightarrow A = 1/4$ $\qquad s=1$ $\quad 5(1/4) + (B+C) = 2 - 5/4 = \frac{3}{4}$

$s=-1$ $\quad 5(1/4) + (-B+C)(-1) = 0$

$$B+C = 3/4$$
$$B-C = -5/4$$
$$2B = -1/2 \Rightarrow B = -1/4$$
$$\Rightarrow C = 1$$

$$\frac{D}{s} + \frac{Es+F}{s^2+4} = \frac{1}{s(s^2+4)}$$

$$D(s^2+4) + (Es+F)(s) = 1$$

$s=0$ $\quad 4D = 1 \Rightarrow D = 1/4$ $\qquad s=1$ $\quad 5(1/4) + E+F = 1 - 5/4$

$$E+F = -1/4$$

$s=-1$ $\quad 5(1/4) + (-E+F)(-1) = 1 \Rightarrow 5/4 + E-F = 1-5/4 \Rightarrow E-F = -1/4$

$$2E = -1/2 \Rightarrow E = -1/4$$
$$\Rightarrow F = 0$$

$$y(t) = 1/4\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}$$
$$+\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} - \mathcal{L}^{-1}\left\{e^{-\pi s}\left(\frac{1}{4}\cdot\frac{1}{s} - \frac{1}{4}\cdot\frac{s}{s^2+4}\right)\right\}$$

$$= \frac{1}{4} - \frac{1}{4}\cos 2t + \frac{1}{2}\sin 2t - \left(\frac{1}{4} - \frac{1}{4}\cos(t-\pi)\right)u(t-\pi)$$

5a. $u_1(t) = \begin{cases} 0 & 0 \le t \le 1 \\ 1 & t > 1 \end{cases}$ $\quad u_3(t) = \begin{cases} 0 & 0 \le t \le 3 \\ 1 & t > 3 \end{cases}$ $\quad u_4(t) = \begin{cases} 0 & 0 \le t \le 4 \\ 1 & t > 4 \end{cases}$

$$u_1(t) + 2u_3(t) - 6u_4(t) = \begin{cases} 0 & 0 \le t \le 1 \\ 1 & 1 < t \le 3 \\ 3 & 3 < t \le 4 \\ -3 & t > 4 \end{cases}$$

(6)

5b. $f(t-3)\,u_3(t) = \begin{cases} 0 & 0\le t<3 \\ \sin t & t>3 \end{cases}$

$f(t) = \sin t$

c. $f(t-1)\,u_2(t) = \begin{cases} 0 & 0\le t\le 2 \\ 2t+2 & t>2 \end{cases}$

$f(t) = 2t$

$f(t-1) = 2(t-1) = 2t-2$

$f(t-2) = 2(t-2)+4-2 = 2(t-2)+2$

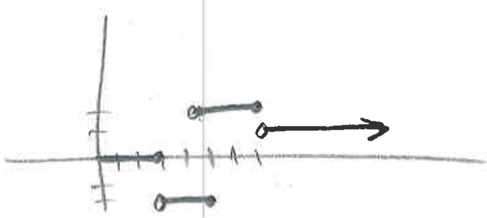
shift must match unit step.

6a. $f(t) = \begin{cases} 0 & 0\le t\le 3 \\ -2 & 3\le t<5 \\ 2 & 5\le t<7 \\ 1 & t>7 \end{cases}$

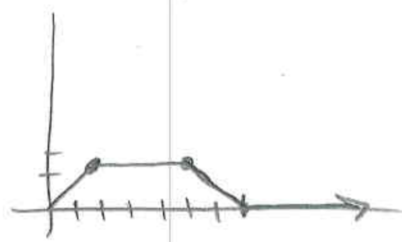
$f(t) = \cancel{0} - 0\,\cancel{u(t-3)} + -2u(t-3) - (-2)u(t-5) + 2u(t-5)$
$\quad -2u(t-7) + 1u(t-7) =$

$-2u(t-3) + 4u(t-5) - 1u(t-7)$

$= -2u_3(t) + 4u_5(t) - u_7(t)$



b. $f(t) = \begin{cases} t & 0\le t<2 \\ 2 & 2\le t<5 \\ 7-t & 5\le t<7 \\ 0 & t\ge 7 \end{cases}$



$f(t) = t - tu(t-2) + 2u(t-2) - 2u(t-5) + (7-(t-5))u(t-5)$
$\quad -(7-(t-5))u(t-7) + 0u(t-7)$

$= t + (2-t)u(t-2) - tu(t-5) + (t-2)u(t-7)$

$= t + (2-t)u_2(t) - tu_5(t) + (t-2)u_7(t)$

$7-(t-5) = 7-t+5 = 2-t$