Math 2255 Homework #2 Key

1. a. $y' = \dfrac{x^2}{y(1+x^3)^4} \Rightarrow \int y\,dy = \int \dfrac{x^2\,dx}{(1+x^3)^4}$ $\quad u = 1+x^3$, $\frac{1}{3}du = \cancel{3}x^2\,dx$

$\frac{1}{2}y^2 = \int \frac{1}{3}u^{-4}\,du = \frac{1}{3}(-\frac{1}{3})u^{-3} = -\frac{1}{9}\frac{1}{(1+x^3)^3} + C$

$y^2 = \dfrac{-2}{9(1+x^3)^3} + C$ $\quad \boxed{y = \pm\sqrt{C - \dfrac{2}{9(1+x^3)^3}}}$

not valid for $y=0$, or $x^3 = -1 \Rightarrow x = -1$

b. $y' = -y^2 \sin x \Rightarrow \int \dfrac{dy}{y^2} = \int -\sin x\,dx \Rightarrow -\dfrac{1}{y} = \cos x + C$

$\boxed{y = \dfrac{-1}{\cos x + C}}$ $\quad$ not valid for $y=0$

c. $y' = (\cos^2 x)(\cos^2 2y) \Rightarrow \int \sec^2 2y\,dy = \int \cos^2 x\,dx = \frac{1}{2}\int 1+\cos 2x\,dx$

$\frac{1}{2}\tan y = \frac{1}{2}x + \frac{1}{4}\sin 2x + C \Rightarrow \tan y = x + \frac{1}{2}\sin 2x + C$

$\boxed{y = \arctan(x + \frac{1}{2}\sin 2x + C)}$ $\quad$ not valid for $y = \frac{(2k+1)\pi}{2}$ (odd multiples of $\frac{\pi}{2}$)

d. $xy' = (1-y^2)^{1/2} \Rightarrow \dfrac{dy}{\sqrt{1-y^2}} = \frac{1}{x}dx \Rightarrow \arcsin y = \ln x + C$

$\arcsin y = \ln Ax \Rightarrow y = \sin(\ln Ax)$ $\quad$ not valid for $y \le -1$ or $y \ge 1$ or $x = 0$

e. $y' = \dfrac{1-2x}{y}$ $\quad y(1) = -2$ $\quad \int y\,dy = \int(1-2x)\,dx$

$\frac{1}{2}y^2 = x - x^2 + C \Rightarrow y^2 = 2x - 2x^2 + C$, $\quad y = \pm\sqrt{2x-2x^2+C}$

use neg root.

$(-2)^2 = 2(1) - 2(1) + C$

$4 = \cancel{2} - \cancel{2} + C \quad C = 4$ $\quad$ not valid for $y=0$

$\boxed{y = -\sqrt{2x - 2x^2 + 4}}$

1. f. (2)

$\sin 2x\,dx + \cos 3y\,dy = 0 \qquad y(\pi/2) = \pi/3$

$\int \cos 3y\,dy = \int -\sin 2x\,dx$

$\frac{1}{3}\sin 3y = \frac{1}{2}\cos 2x + C$

$\sin 3y = \frac{3}{2}\cos 2x + C$

$\frac{3}{2}\cos(\pi) + C = \sin(\pi)$

$\frac{3}{2}(-1) + C = 0$

$C = 3/2$

$\sin 3y = \frac{3}{2}\cos 2x + \frac{3}{2}$

$3y = \arcsin\left[\frac{3}{2}(\cos 2x + 1)\right]$

$\boxed{y = \frac{1}{3}\arcsin\left[\frac{3}{2}(\cos 2x + 1)\right]}$

g. $y' = ty(4-y) \qquad y(0) = y_0$

$\int \frac{dy}{y(4-y)} = \int t\,dt \Rightarrow \int \frac{A}{y} + \frac{B}{y-4}\,dy = \int t\,dt \Rightarrow A\ln y + B\ln|y-4| = \frac{1}{2}t^2 + C$

$A(y-4) + By = 1$

$y=0 \Rightarrow -4A = 1 \quad A = -\frac{1}{4}$

$y=4 \Rightarrow 4B = 1 \quad B = \frac{1}{4}$

$\frac{1}{4}\ln|y-4| - \frac{1}{4}\ln|y| = \frac{1}{2}t^2 + C$

$\ln|y-4| - \ln|y| = 2t^2 + C$

$\ln\left|\frac{y-4}{y}\right| = 2t^2 + C$

$1 - \frac{4}{y} = e^{2t^2+C} = Ae^{2t^2} - 1$

$\frac{4}{y} = 1 - Ae^{2t^2} \Rightarrow \frac{y}{4} = \frac{1}{1-Ae^{2t^2}}$

$y = \frac{4}{1-Ae^{2t^2}}$

$y_0 = \frac{4}{1-Ae^{2(0)^2}} = \frac{4}{1-A}$

$\frac{1}{y_0} = \frac{1-A}{4} \Rightarrow \frac{4}{y_0} = 1-A$

$\frac{4}{y_0} - 1 = -A \Rightarrow A = 1 - \frac{4}{y_0}$

$\boxed{y = \frac{4}{1-\left(1-\frac{4}{y_0}\right)e^{2t^2}}}$

not valid for $y=0, y=4$

2. $\frac{dS}{dt} = rS + k = .1S + k$ $\quad S(0) = 8000$ (3)

$S(3) = 0$

$S' = r(S + \frac{k}{r}) \Rightarrow \int \frac{dS}{S + \frac{k}{r}} = \int r\,dt \Rightarrow \ln|S + \frac{k}{.1}| = .1t + C$

$S + 10k = e^{.1t+C} = Ae^{.1t} \Rightarrow S(t) = A_0 e^{.1t} - 10k$

$8000 = A_0 - 10k$

$0 = A_0 e^{.3} - 10k \Rightarrow \frac{10k}{10} = \frac{A_0 e^{.3}}{10} \Rightarrow k = \frac{A_0 e^{.3}}{10}$

$8000 = A_0 - \cancel{10}\left(\frac{A_0 e^{.3}}{\cancel{10}}\right) = A_0 - A_0 e^{.3} = A_0(1 - e^{.3})$

$A_0 = \frac{8000}{1 - e^{.3}} = -22866.37$ (dollars & cents)

$k = \frac{A_0 e^{.3}}{10} = = -3086.64$ The signs are negative because you are paying off a loan (and decreasing the value) rather than a savings account (& increasing the value).

So payments are $\approx$ \$3086.64 per year.

interest can be calculated by total payments

$S(t) = -22866.37 e^{.1t} + 30,866.47$

$\underset{\text{paid}}{3086.64 \times 3 = 9259.92} - \underset{\text{loan}}{8000} = \$1259.92$ interest.

3. $\frac{dy}{dt} = \frac{(.05 + \sin t)y}{5} \Rightarrow \frac{dy}{y} = \frac{1}{5}(.05 \sin t)\,dt$ $\quad y(0) = 1$

$\ln|y| = \frac{1}{5}(.05t - \cos t) + C \Rightarrow y = Ae^{.01t - \cos t}$ $\quad 1 = Ae^{0-1} \Rightarrow A = e$

$y = e^{.01t - \cos t + 1}$ first passes 2 at $\tau \approx 1.245$ time units

3 Cont'd

Suppose $y(0)=5$ (4)

$5 = Ae^{0-1} \Rightarrow A = 5e \Rightarrow y = 5e^{.01t-\cos t+1}$

$\tau = 1.24579\ldots$ same value as before.

4. u = surroundings T = temperature t = time

$\frac{dT}{dt} = k(T-u)$ $T(0) = 200°$ $u = 70°$

$T(1) = 190$

$\int \frac{dT}{T-70} = \int k\,dt \Rightarrow \ln|T-70| = kt + C \Rightarrow T-70 = T_0 e^{kt}$

$\Rightarrow T(t) = T_0 e^{kt} + 70$ $200 = T_0 + 70 \Rightarrow T_0 = 130$

$T(t) = 130e^{kt} + 70$ $190 = 130e^{k(1)} + 70 \Rightarrow$

$\frac{120}{130} = e^k \Rightarrow \ln\left(\frac{12}{13}\right) = k \approx -.08$

$T(t) = 130e^{-.08t} + 70$

$150 = 130e^{-.08t} + 70 \Rightarrow \frac{80}{130} = \frac{\cancel{130}}{\cancel{130}} e^{-.08t} \Rightarrow \ln\left(\frac{8}{13}\right) = -.08t$

$\Rightarrow t = \frac{\ln\left(\frac{8}{13}\right)}{-.08} \approx 6.0656\ldots \approx 6$ minutes

5. a) $|v|$ = speed call $|v| = s$ $s(0) = 0$

$\frac{mg}{m} - \frac{ks}{m} = \frac{ms'}{m} \Rightarrow s' = g - \frac{k}{m}s$ $g = 32$

$s' = 32 - \frac{.75}{5.625}s$ Convert pounds to stones: $\frac{180}{32} =$

$s' = 32 - \frac{2}{15}s \Rightarrow s' = -\frac{2}{15}(s-240)$

$\int \frac{ds}{s-240} = \int -\frac{2}{15}\,dt \Rightarrow \ln|s-240| = -\frac{2}{15}t + C \Rightarrow s - 240 = Ae^{-2/15 t}$

$s(t) = Ae^{-2/15 t} + 240$ $0 = A\cancel{e^0} + 240 \Rightarrow A = -240$

$s(t) = 240 - 240Ae^{-2/15 t}$ good until $t = 10$

5 cont'd

$\boxed{S(10) = 176.74}$

S(10) acts like s(0) for second piece

c) $s' = 32 - \frac{12}{5.625} s \Rightarrow s' = 32 - \frac{32}{15} s = -\frac{32}{15}(s-15)$

$\int \frac{ds}{s-15} = \int -\frac{32}{15}\, dt \Rightarrow \ln|s-15| = -\frac{32}{15}t + C \Rightarrow s(t) = Ae^{-32/15} + 15$

$S(10) = 176.74 = A\cancel{e^{0}} + 15 \qquad 161.74 = A$

$s(t) = 161.74e^{-32/15t} + 15 \qquad$ for $t > 10$

limiting velocity w/ chute is $\boxed{15 \text{ ft/sec.}}$

b) to compute to distance fallen integrate velocity [0,10]

$\int_0^{10} 240 - 240e^{-2/15t}\, dt = 240t + 1800e^{-2/15t}\Big|_0^{10}$

$240(10) + 1800e^{-\frac{20}{15}} - 0 - 1800 = \boxed{1074.47 \text{ feet.}}$

6. a. $(t-3)y' + \ln(t)y = 2t \qquad y(1) = 2$

$y' + \frac{\ln(t)}{(t-3)}y = \frac{2t}{t-3} \qquad p(t) = \frac{\ln(t)}{t-3}$ defined on $(0,3) \cup (3,\infty)$

$\boxed{t = 1 \text{ on } (0,3)}$

b. $(4-t^2)y' + 2ty = 3t^2 \qquad y(-3) = 1$

$y' + \frac{2t}{(2-t)(2+t)}y = \frac{3t^2}{(2-t)(2+t)} \qquad p(t) = \frac{2t}{(2-t)(2+t)}$ defined on $(-\infty,-2) \cup (-2,2) \cup (2,\infty)$

$\boxed{t = -3 \text{ on } (-\infty,-2)}$

c. $y' = (1-t^2-y^2)^{1/2} \qquad f(t,y) = (1-t^2-y^2)^{1/2}$

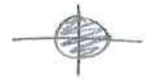
$\frac{\partial f}{\partial y} = \frac{1}{\cancel{2}}(1-t^2-y^2)^{-1/2} \cdot (-\cancel{2}y) =$

$= \frac{-y}{\sqrt{1-t^2-y^2}}$ This will remove the equal sign in our original inequality

$1 - t^2 - y^2 \geq 0 \Rightarrow 1 \geq t^2 + y^2$

$t^2 + y^2 \leq 1$

This is the unit circle



6c cont'd

allowable region anywhere inside circle but not on the edge.

d. $\dfrac{dy}{dt} = \dfrac{(\cot t)y}{1+y}$

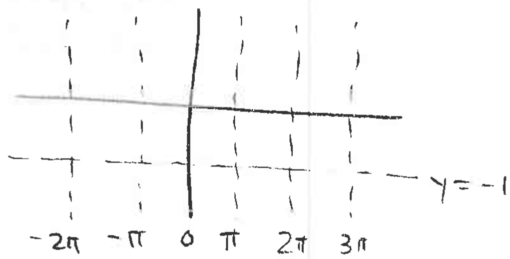
$\cot t$ not defined at multiples of π

$1+y \neq 0 \Rightarrow y = -1$

$\dfrac{\partial f}{\partial y} = (\cot t)\left[\dfrac{1(1+y) - 1(y)}{(1+y)^2}\right] =$

$(\cot t)\left[\dfrac{1}{(1+y)^2}\right]$

no new conditions



ok everywhere except at dotted lines

7. $\dfrac{t^2y' + 2ty - y^3 = 0}{t^2}$ $\qquad y' + \dfrac{2}{t}y = y^3$ $\qquad n=3$ $\quad (1-n)y^{-n} = -2y^{-3}$

$-2y^{-3}y' - \dfrac{4}{t}y^{-2} = -2$ $\qquad$ let $z = y^{-2}$ $\quad z' = -2y^{-3}y'$

$z' - \dfrac{4}{t}z = -2$ $\qquad \mu = e^{\int -\frac{4}{t}dt} = e^{-4\ln t} = e^{\ln t^{-4}} = t^{-4}$

$t^{-4}z' - 4t^{-5}z = -2t^{-4}$

$\int (t^{-4}z)' = \int -2t^{-4}dt \Rightarrow t^{-4}z = \left.\dfrac{+2t^{-3}}{+3} + c\right)t^4$

$\dfrac{2}{3}t + Ct^4 = z = y^{-2} \Rightarrow y^2 = \dfrac{1}{\frac{2}{3}t + Ct^4} \Rightarrow \boxed{y = \pm\sqrt{\dfrac{1}{\frac{2}{3}t + Ct^4}}}$

8. a. $M = 3x^2 - 2xy + 2 \qquad M_y = -2x$ ✓
$N = 6y^2 - x^2 + 3 \qquad N_x = -2x$

$\int 3x^2 - 2xy + 2\,dx = x^3 - x^2y + 2x + f(y)$

$\int 6y^2 - x^2 + 3\,dy = 2y^3 - x^2y + 3y + g(x)$

$\psi(x,y): x^3 + 2y^3 - x^2y + 2x + 3y = C$

8b. $M = 9x^2 + y - 1 \quad M_y = 1$

$N = -(4y - x) = x - 4y \quad N_x = 1$ ✓

(7)

$y(1) = 0$

$x = 1,\ y = 0$

$\int 9x^2 + y - 1\, dx = 3x^3 + xy - x + f(y)$

$\int x - 4y\, dy = xy - 2y^2 + g(x)$

$\psi(x,y):\ 3x^3 + xy - x - 2y^2 = c$

$3(1)^3 + (1)(0) - (1) - 2(0) = c \Rightarrow 3 - 1 = 2 = c$

$\boxed{3x^3 + xy - x - 2y^2 = 2}$

c. $M = ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x$

$M_y = e^{xy}\cos 2x + xye^{xy}\cos 2x - 2xe^{xy}\sin 2x$

$N = xe^{xy}\cos 2x - 3$

$N_x = e^{xy}\cos 2x + xye^{xy}\cos 2x - 2xe^{xy}\sin 2x$ ✓

for the $e^{xy}\cos 2x$, etc. terms

$\frac{d}{dx}[e^{xy}\cos 2x] = ye^{xy}\cos 2x - 2e^{xy}\sin 2x$ (compare w/M)

$\frac{d}{dy}[e^{xy}\cos 2x] = xe^{xy}\cos 2x$ (compare w/ N)

accounts for all (x,y) terms. integrate for x-only & y-only terms

$\psi(x,y):\ e^{xy}\cos 2x + x^2 - 3y = c$

9. a. $(x^2y^3)dx + x(1+y^2)dy = 0\Big]\frac{1}{xy^3} \Rightarrow x\,dx + \frac{1+y^2}{y^3}dy = 0$

$(\mu M)_y = 0 \quad (\mu N)_x = 0$

$\int x\,dx = \frac{1}{2}x^2 + f(y) \quad \int y^{-3} + \frac{1}{y}\,dy = \frac{y^{-2}}{-2} + \ln y + g(x)$

$= -\frac{1}{2y^2} + \ln y + g(x)$

$\psi(x,y):\ \boxed{\frac{1}{2}x^2 - \frac{1}{2y^2} + \ln y = c}$

9b. $y\,dx + (2x - ye^y)dy = 0$ 8

$\mu = y$

$y^2 dx + (2xy - y^2 e^y)\,dy = 0$

$(\mu M)_y = 2y \qquad (\mu N)_x = 2y$ ✓

$\int y^2 dx = xy^2 + f(y) \qquad \int 2xy - y^2 e^y \, dy =$

$= xy^2 - y^2 e^y + 2ye^y - 2e^y + g(x)$

$\pm$	u	dv
$-$ $+$	y^2	e^y
$+$ $-$	$2y$	e^y
$-$ $+$	2	e^y
$+$ $-$	0	e^y

$\psi(x,y)$: $\boxed{xy^2 - y^2 e^y + 2ye^y - 2e^y = C}$

c. $dx + (\frac{x}{y} - \sin y)dy = 0$

$M_y = 0 \qquad N_x = \frac{1}{y} \qquad \frac{M_y - N_x}{N} = \frac{0 - \frac{1}{y}}{\frac{x}{y} - \sin y}$ ✗

$\frac{N_x - M_y}{M} = \frac{\frac{1}{y} - 0}{1} = \frac{1}{y} \qquad \frac{d\mu}{dy} = \frac{1}{y}\mu \Rightarrow \int \frac{d\mu}{\mu} = \int \frac{1}{y} dy \Rightarrow$

$\ln \mu = \ln y \Rightarrow \mu = y$

$y\,dx + (x - y\sin y)\,dy = 0$

$(\mu M)_y = 1 \qquad (\mu N)_x = 1$ ✓

$\int y\,dx = xy + f(y) \qquad \int x - y\sin y\,dy$

$= xy + y\cos y - \sin y + g(x)$

$\pm$	u	dv
$-$ $+$	y	$\sin y$
$+$ $-$	1	$-\cos y$
$-$ $+$	0	$-\sin y$

$\psi(x,y)$: $\boxed{xy + y\cos y - \sin y = C}$

d. $e^x dx + (e^x \cot y + 2y \csc y)\,dy = 0$

$M_y = 0 \qquad N_x = e^x \cot y$

$\frac{M_y - N_y}{N} = \frac{0 - e^x \cot y}{e^x \cot y + 2y \csc y}$ ✗

9d cont'd

$$\frac{N_x - M_y}{M} = \frac{\cancel{e^x}\cot y - 0}{\cancel{e^x}} = \cot y \qquad \frac{d\mu}{dy} = \cot y \cdot \mu$$

$$\Rightarrow \int \frac{d\mu}{\mu} = \int \cot y \, dy \Rightarrow \ln \mu = \ln \sin y \Rightarrow \mu = \sin y$$

$$e^x \sin y \, dx + (e^x \cos y + 2y)\,dy = 0$$

$$(\mu M)_y = e^x \cos y \qquad (\mu N)_x = e^x \cos y \quad \checkmark$$

$$\int e^x \sin y \, dx = e^x \sin y + f(y)$$

$$\int e^x \cos y + 2y \, dy = e^x \sin y + y^2 + g(x)$$

$$\psi(x,y): \boxed{e^x \sin y + y^2 = c}$$