Math 2255, Final Exam, Spring 2013 Name KEY

Instructions: Show all work. Answers with no work will be graded all or nothing unless the point of the problem is to show the work (in which case, no work will receive no credit). Use exact values (fractions and square roots, etc.) unless the problem tells you to round, is a word problem, or begins with decimal values.

1. Solve the differential equation $y'' + (x^3+1)y = 0$ by methods of series solutions. Be sure to list at least 4 terms for each unknown in your solution. (35 points)

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + x^3\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+3} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=-3}^{\infty} a_{n+5}(n+5)(n+4)x^{n+3} + \sum_{n=0}^{\infty} a_n x^{n+3} + \sum_{n=-3}^{\infty} a_{n+3}x^{n+3} = 0$$

$$a_2(2)(1)x^0 + a_3(3)(2)x^1 + a_4(4)(3)x^2 + \sum_{n=0}^{\infty} a_{n+5}(n+5)(n+4)x^{n+3}$$

$$+\sum_{n=0}^{\infty} a_n x^{n+3} + a_0x^0 + a_1x^1 + a_2x^2 + \sum_{n=0}^{\infty} a_{n+3}x^{n+3} = 0$$

$$\sum_{n=0}^{\infty}\left[a_{n+5}(n+5)(n+4) + a_n + a_{n+3}\right] = 0$$

$$a_{n+5} = \frac{-a_n}{(n+5)(n+4)} - \frac{a_{n+3}}{(n+5)(n+4)}$$

$2a_2 + a_0 = 0 \Rightarrow a_2 = -\frac{1}{2}a_0$

$6a_3 + a_1 = 0 \Rightarrow a_3 = -\frac{1}{6}a_1$

$12a_4 + a_2 = 0 \Rightarrow a_4 = -\frac{1}{12}a_2 = -\frac{1}{12}\left(-\frac{1}{2}a_0\right) = \frac{1}{24}a_0$

$n=0$

$$a_5 = \frac{-a_0}{(5)(4)} - \frac{a_3}{5\cdot 4} = -\frac{1}{20}a_0 - \frac{1}{20}\left(-\frac{1}{6}a_1\right) = -\frac{1}{20}a_0 + \frac{1}{120}a_1$$

$n=1$

$$a_6 = \frac{-a_1}{6\cdot 5} - \frac{a_4}{6\cdot 5} = -\frac{1}{30}a_1 - \frac{1}{30}\left(\frac{1}{24}a_0\right) = -\frac{1}{30}a_1 - \frac{1}{720}a_0$$

$n=2$

$$a_7 = \frac{-a_2}{7\cdot 6} - \frac{a_5}{7\cdot 6} = -\frac{1}{42}\left(-\frac{1}{2}a_0\right) - \frac{1}{42}\left(-\frac{1}{20}a_0 + \frac{1}{120}a_1\right) = \frac{1}{84}a_0 + \frac{1}{84}a_0 - \frac{1}{5040}a_1 = \frac{1}{42}a_0 - \frac{1}{5040}a_1$$

$$y = a_0\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{20}x^5 - \frac{1}{720}x^6 + \frac{1}{42}x^7 + \dots\right) + a_1\left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{30}x^6 - \frac{1}{5040}x^7 + \dots\right)$$

2. Determine if the equations contain any singular points and whether they are regular or irregular. (8 points each)

a. $x(x-1)y'' + 6x^2y' + 3y = 0$

$$y'' + \frac{6x^{\cancel{2}}}{\cancel{x}(x-1)}y' + \frac{3}{x(x-1)}y = 0$$

$x=0,\ x=1$ regular Singular points

b. $(x+1)^2y'' + 3(x^2-1)y' + 3y = 0$

$$y'' + \frac{3(x-1)^2\cancel{(x+1)^2}}{\cancel{(x+1)^2}}y' + \frac{3}{(x+1)^2}y = 0$$

$x=-1$ regular singlala point

c. $x \sin x\, y'' + 3y' + xy = 0$

$$y'' + \frac{3}{x\sin x}y' + \frac{\cancel{x}}{\cancel{x}\sin x}y = 0$$

$x=0, \pi, \ldots k\pi$ irregular Singular points

3. Solve the differential equation $x^2y' + 3xy' + y = 0$. (17 points)

$n(n-1) + 3n + 1 =$

$n^2 - n + 3n + 1 = n^2 + 2n + 1 = (n+1)^2 = 0$

$n = -1$ repeated

$y = c_1 x^{-1} + c_2 x^{-1} \ln x$

4. For the following differential equations, determine the order of the equation, whether it is linear or non-linear, and whether it is ordinary or partial. (6 points each)

   a. $y\ddot{y} + ty^2 = 0$ 2nd order, ordinary, linear

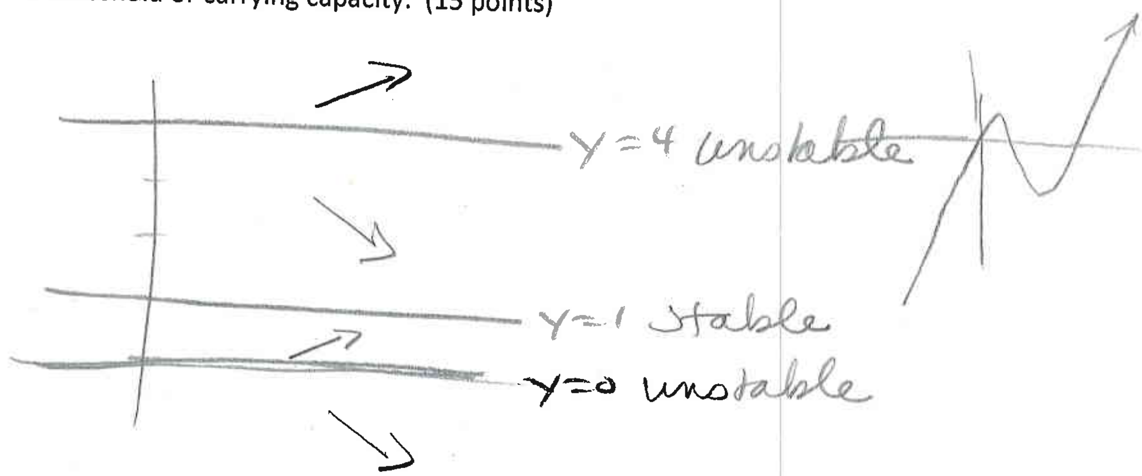
   $\ddot{y} + ty = 0$

   b. $t^2 \frac{d^2y}{dt^2}\frac{dy}{dt} + t\frac{dy}{dt} + 2y = sint$ 2nd order ordinary, non-linear

   c. $\frac{(x^2 + y)dx}{2-x} + \frac{(2 - x)dy}{(2-x)\,dx} = 0$ linear, ordinary, first order

   $y' + \frac{x^2+y}{2-x} = 0 = y' + \frac{y}{2-x} = \frac{x^2}{x-2}$

5. Sketch the direction field for the graph $y' = y(y-1)(y-4)$. State whether each equilibrium is stable, unstable or semi-stable, and for each equilibrium greater than zero, state whether it's a threshold or carrying capacity. (15 points)



y=1 carrying capacity y=4 threshold.

6. For the following equations, determine the solution method (integrating factor/linear, separable equations, homogeneous (substitution y=vx), Bernoulli equations, exact equation (check for integrating factor if need be) or if the problem must be done numerically. Do not solve the equations. (7 points each)

   a. $6xydx + (4y + 9x^2)dy = 0$ exact (needs integrating factor)

b. $y' = t + 2y$ linear/integrating factor

c. $yy' + \frac{1}{x}y^2 = x\sqrt{y}$ bernoulli's

d. $xy' = (1 - y^2)^{1/2}$ Separation of variables

7. Solve the linear differential equation $y' + xy = x$ ~~either~~ by integrating factor. (20 points)

$$e^{\int x\,dx} = e^{x^2/2}$$

$$e^{x^2/2}y' + xe^{x^2/2}y = xe^{x^2/2}$$

$$\int \left(e^{x^2/2}y\right)' = \int xe^{x^2/2}$$

$$e^{x^2/2}y = e^{x^2/2} + C_1$$

$$y = 1 + C_1e^{-x^2/2}$$

8. Solve the following second order homogeneous linear differential equations for the general solution. (15 points each)

a. $y'' - 4y' - 12y = 0$

$$r^2 - 4r - 12 = 0$$
$$(r-6)(r+2) = 0$$
$$r = 6, -2$$
$$y = c_1 e^{6t} + c_2 e^{-2t}$$

b. $y'' + 10y' + 25y = 0$

$$r^2 + 10r + 25 = 0$$
$$(r+5)^2 = 0$$
$r = -5$ repeated
$$y = c_1 e^{-5t} + c_2 t e^{-5t}$$

9. Use the method of variation of parameters or undetermined coefficients and your solution from problem 8a to find the particular solution to $y'' - 4y' - 12y = te^{-2t}$. (20 points)

$$W = \begin{vmatrix} e^{6t} & e^{-2t} \\ 6e^{6t} & -2e^{-2t} \end{vmatrix} = -2e^{4t} - 6e^{4t} = -8e^{4t}$$

$$y = -e^{6t}\int \frac{te^{-2t} \cdot e^{-2t}}{-8e^{4t}}\,dt + e^{-2t}\int \frac{te^{-2t}\,\cancel{e^{6t}}}{-8\cancel{e^{4t}}}\,dt$$

$$-\frac{1}{8}e^{6t}\int te^{-8t}\,dt \quad -\frac{1}{8}e^{-2t}\int t\,dt =$$

±	u	dv
+	t	e^{-8t}
−	1	$-\frac{1}{8}e^{-8t}$
+	0	$\frac{1}{64}e^{-8t}$

$$-\frac{1}{8}\left[e^{6t}\left(-\frac{1}{8}te^{-8t} - \frac{1}{64}e^{-8t}\right) - e^{-2t}\left(\frac{t^2}{2}\right)\right] + c_1 e^{6t} + c_2 e^{-2t}$$

$$\frac{1}{64}te^{-2t} + \frac{1}{512}e^{-2t} + \frac{1}{16}t^2e^{-2t} + c_1e^{6t} + c_2e^{-2t}$$

$$\frac{1}{16}t^2e^{-2t} + \frac{1}{64}te^{-2t} + c_1e^{6t} + c_2e^{-2t}$$

10. Set up the Wronskian matrix, but do not calculate it, for the problem in 8b. What does Abel's theorem tell you its value should be? (15 points)

$y_1 = e^{-5t}$

$y_2 = te^{-5t}$

$$W = \begin{vmatrix} e^{-5t} & te^{-5t} \\ -5e^{-5t} & e^{-5t} - 5te^{-5t} \end{vmatrix}$$

$$W = e^{-\int 10\, dt} = c_1 e^{-10t}$$

11. Find the Laplace transform for the following functions. You may use the attached table of transform formulas. (28 points)

a. $f(t) = te^{-2t}$

$$\frac{1}{(s+2)^2}$$

b. $f(t) = \cosh 7t$

$$\frac{s}{s^2-49}$$

c. $f(t) = \int_0^t (t-\tau)^5 \sin\tau \, d\tau$

$$\frac{120}{s^6} \cdot \frac{1}{s^2+1}$$

d. $f(t) = e^{-t} sin4t$

$$\frac{4}{(s+1)^2+16}$$

12. Use the Laplace transform formulas to find the inverse Laplace transform of the following functions in s. (21 points)

a. $F(s) = \frac{1}{s+2}$

$$e^{-2t}$$

b. $F(s) = \frac{3s-7}{s^2+81}$

$$\frac{3s}{s^2+81} - \frac{7}{s^2+81} \Rightarrow 3\cos 9t - \frac{7}{9}\sin 9t$$

c. $F(s) = \frac{s+6}{s^2+2s-3}$

$$= \frac{s+6}{(s^2+2s+1)-4} = \frac{s+6}{(s+1)^2-4} = \frac{(s+1)}{(s+1)^2-4} + \frac{5}{(s+1)^2-4}$$

$$e^{-t}\cosh 2t + \frac{5}{2}e^{-t}\sinh 2t$$

OR

$$\frac{s+6}{s^2+2s-3} = \frac{s+6}{(s+3)(s-1)} = \frac{A}{s+3} + \frac{B}{s-1} = Ae^{-3t} + Be^{t}$$

$$= -\frac{3}{4}e^{-3t} + \frac{7}{4}e^{t}$$

$$A(s-1) + B(s+3) = s+6$$

$s=1$ $\quad 4B = 7 \quad B = \frac{7}{4}$

$s=-3$ $\quad -4A = 3 \quad A = -\frac{3}{4}$

13. A mass weighing 12 lbs stretches a spring 18 inches. If the mass is pushed upwards, contracting the spring a distance of 3 inch and then set in motion with a downward velocity of 5 ft/sec, and if there is no damping, and the system is driven by a force of 2cos(3t) pounds, set up the differential equation that models the system. You can earn up to five additional points for solving the complete equation. (15 points)

$m = \frac{12}{32} = \frac{3}{8}$

$12 = k\,\frac{18}{12}$

$k = 12 \cdot \frac{2}{3} = 8$

$y(0) = +1/4$

$y'(0) = -5$

$8\left(\frac{3}{8}y'' + 8y = 2\cos 3t\right.$

$3y'' + 64y = 16\cos 3t$

$3r^2 + 64 = 0$

$r = \pm\frac{8}{\sqrt{3}}i$

$Y_c = C_1\cos\left(\frac{8}{\sqrt{3}}t\right) + C_2\sin\left(\frac{8}{\sqrt{3}}t\right)$

$Y = A\cos 3t + B\sin 3t$

$Y' = -3A\sin 3t + 3B\cos 3t \qquad Y'' = -9A\cos 3t - 9B\sin 3t$

$3(-9A\cos 3t - 9B\sin 3t) + 64(A\cos 3t + B\sin 3t) = 16\cos 3t$

$(-27+64)A\cos 3t + (-27+64)B\sin 3t = 16\cos 3t$

$37A\cos 3t = 16\cos 3t \qquad B = 0$

$37A = 16$

$A = \frac{16}{37}$

$y(t) = C_1\cos\left(\frac{8}{\sqrt{3}}t\right) + C_2\sin\left(\frac{8}{\sqrt{3}}t\right) + \frac{16}{37}\cos 3t$

Formulas for Laplace Transforms
Math 255 Summer 2012

General:

1. $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$
2. $\mathcal{L}\{f^n(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$
3. $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$
4. $\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as} F(s)$
5. $\mathcal{L}\{g(t)\mathcal{U}(t-a)\} = e^{-as} \mathcal{L}\{g(t+a)\}$
6. $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[F(s)]$
7. $f * g = \int_0^t f(\tau) g(t-\tau) d\tau$
8. $\mathcal{L}\{f * g\} = F(s)G(s)$
9. $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$
10. $\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)dt$ (f is periodic)

Specific Functions:

1. $\mathcal{L}\{1\} = \frac{1}{s}$
2. $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$, n=1,2,3...
3. $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$
4. $\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}$
5. $\mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2}$
6. $\mathcal{L}\{\sinh kt\} = \frac{k}{s^2-k^2}$
7. $\mathcal{L}\{\cosh kt\} = \frac{s}{s^2-k^2}$
8. $\mathcal{L}\{\mathcal{U}(t-a)\} = \frac{e^{-as}}{s}$

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13 contd $\quad 1/4 = c_1 + \frac{16}{37} \implies c_1 = \frac{-27}{148}$

$y'(t) = -\frac{8}{\sqrt{3}} c_1 \sin\left(\frac{8}{\sqrt{3}}t\right) + \frac{8}{\sqrt{3}} c_2 \cos\left(\frac{8}{\sqrt{3}}t\right) + \frac{-16}{37}(3)\sin 3t$

$-\frac{54}{37\sqrt{3}} \cancel{\sin\left(\frac{8}{\sqrt{3}}t\right)} + \frac{8}{\sqrt{3}} c_2 \cos\left(\frac{8}{\sqrt{3}}t\right) + \cancel{\frac{-48}{37}\sin 3t}$

$-5 = \frac{8}{\sqrt{3}} c_2 \cancel{\cos\left(\frac{8}{\sqrt{3}}t\right)}$

$c_2 = \frac{-5\sqrt{3}}{8}$

$\boxed{y(t) = \frac{-27}{148}\cos\left(\frac{8}{\sqrt{3}}t\right) - \frac{5\sqrt{3}}{8}\sin\left(\frac{8}{\sqrt{3}}t\right) + \frac{16}{37}\cos 3t}$