Name KEY

Math 2255, Exam #3, Spring 2013

Instructions: Show all work. Answers with no work will be graded all or nothing unless the point of the problem is to show the work (in which case, no work will receive no credit). Use exact values (fractions and square roots, etc.) unless the problem tells you to round, is a word problem, or begins with decimal values.

1. Use the **definition** of Laplace transforms to find the transform F(s) for the function $f(t) = te^t$. You may check your answers with the included table, but you must show the integration work to receive credit. (15 points)

$$\int_0^\infty e^{-st} t e^t \, dt = \int_0^\infty t e^{-(s-1)t} \, dt$$

$$= -\frac{t}{s-1} e^{-(s-1)t} - \frac{1}{(s-1)^2} e^{-(s-1)t} \Big|_0^\infty$$

$$0 - 0 + 0 + \frac{1}{(s-1)^2} = \frac{1}{(s-1)^2}$$

+/-	u	dv
+	t	$e^{-(s-1)t}$
-	1	$-\frac{1}{s-1} e^{-(s-1)t}$
+	0	$\frac{1}{(s-1)^2} e^{-(s-1)t}$

2. Find the Laplace transform for the following functions. You may use the attached table of transform formulas. (20 points)

a. $f(t) = te^{4t}$

$$\frac{1}{(s-4)^2}$$

b. $f(t) = \cosh 4t$

$$\frac{s}{s^2-16}$$

c. $f(t) = \int_0^t (t-\tau)\sin\tau \, d\tau$

$\frac{1}{s^2} \cdot \frac{1}{s^2+1}$

d. $f(t) = \begin{cases} 2t+1, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$

$(2t+1) - (2t+1)u(t-1)$

$2(t+1)+1 = 2t+3$

$\frac{2}{s^2} + \frac{1}{s} - e^{-s}\left(\frac{2}{s^2} + \frac{3}{s}\right)$

e. $f(t) = e^t \sin 3t$

$\frac{3}{(s-1)^2+9}$

3. Use the Laplace transform formulas to find the inverse Laplace transform of the following functions in s. (20 points)

a. $F(s) = \frac{1}{4s+1}$ $= \frac{1}{4}\left(\frac{1}{s+\frac{1}{4}}\right) \Rightarrow \frac{1}{4}e^{-\frac{1}{4}t}$

b. $F(s) = \frac{5s-11}{s^2+9}$ $= \frac{5s}{s^2+9} - \frac{11}{s^2+9} \Rightarrow$

$5\cos 3t - \frac{11}{3}\sin 3t$

c. $F(s) = \frac{s}{s^2+2s-3}$ (use partial fractions for this one) $= \frac{A}{s+3} + \frac{B}{s-1}$

$(s+3)(s-1)$

$s = A(s-1) + B(s+3)$

$s=1 \quad 1 = 4B \Rightarrow B = 1/4$ $\quad s=-3 \quad -3 = -4A$

$A = 3/4$

$\Rightarrow \frac{3}{4}e^{-3t} + \frac{1}{4}e^{t}$

d. $F(s) = \frac{1}{s^2+2s+5}$ $= \frac{1}{2}e^{-t}\sin 2t$

$s^2+2s+1+4$

$(s+1)^2+4$

e. $F(s) = \frac{5}{s^4+49s^2}$ (use convolutions for this one) $= \frac{5}{s^2} \cdot \frac{1}{s^2+49}$

$5t * \frac{1}{7}\sin 7t$ or

$\int_0^t \frac{5}{7}(t-\tau)\sin 7\tau \, d\tau$

4. Use Laplace Transforms to solve the given initial value problem $y' + 6y = e^{4t}, y(0) = 2$. Solve this one completely for y(t) including all constants and all integrations. (15 points)

$$sY(s) - \cancel{2} + 6Y(s) = \frac{1}{s-4} + 2$$

$$Y(s)\cancel{(s+6)} = \frac{1 + 2(s-4)}{(s-4)(s+6)} = \frac{2s-7}{(s-4)(s+6)}$$

$$\frac{A}{s-4} + \frac{B}{s+6} \Rightarrow 2s-7 = A(s+6) + B(s-4)$$

$s = 4 \qquad 1 = 10A \Rightarrow A = 1/10$

$s = -6 \qquad -19 = B(-10) \Rightarrow B = \frac{19}{10}$

$$y(t) = 1/10\, e^{4t} + \frac{19}{10} e^{-6t}$$

5. For each of the following differential equations (or integro-differential equations) set up to solve by Laplace transforms. Find Y(s). You do not need to perform the inverse transform at the end. (10 points each)

a. $6y'' - 5y' + y = e^{t} \cos 3t, y(0) = 4, y'(0) = 0$.

$$6s^2Y(s) - 6s(4) - 5sY(s) - 5(4) + Y(s) = \frac{s-1}{(s-1)^2+9}$$

$$Y(s)(6s^2 - 5s + 1) = \frac{s-1}{(s-1)^2+9} + 24s + 20$$

$s^2 - 2s + 10$

$$Y(s) = \frac{s-1}{(s^2-2s+10)(6s^2-5s+1)} + \frac{24s+20}{6s^2-5s-1}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

b. $y''' - 8y = f(t), y(0) = 1,\ y'(0) = -1, y''(0) = 0, f(t) = \begin{cases} e^t \sin t, & 0 \le t < \pi \\ \sin t, & t \ge \pi \end{cases}$

$s^3Y(s) - s^2 + s - 0 - 8Y(s) =$ $\qquad e^t\sin t - e^t \sin t\, u(t-\pi) - \sin(t-\pi)\, u(t-\pi)$

$\frac{1}{(s-1)^2+1} + e^{\pi} \cdot e^{-\pi s}\left(\frac{1}{(s-1)^2+1} - \frac{1}{s^2+1}\right)$ $\qquad e^{t+\pi}\sin(t+\pi) = -e^{\pi} \cdot e^t \sin t$

$Y(s)(s^3-8) = \frac{1}{(s-1)^2+1} + s^2 - s + e^{\pi}e^{-\pi s}\left(\frac{1}{(s-1)^2+1} - \frac{1}{s^2+1}\right)$

$s^2-2s+2 \qquad s^2-2s+2$

$\frac{s^4-3s^3-2s+1}{s^2-2s+2} \qquad \frac{\cancel{s^2}+1-\cancel{s^2}+2s+2}{(s^2-2s+2)(s^2+1)}$

$Y(s) = \frac{s^4-3s^3-2s+1}{(s^2-2s+2)(s^3-8)} + e^{\pi}e^{-\pi s}\left(\frac{2s+3}{(s^2-2s+2)(s^2+1)(s^3-8)}\right)$

c. $y'(t) + 4y(t) - \int_0^t \cos(\tau)\, y(t-\tau)d\tau = 1$ $\quad y(0) = 0$

$sY(s) + 4Y(s) - \frac{s}{s^2+1}Y(s) = \frac{1}{s}$

$Y(s)\left(s + 4 - \frac{s}{s^2+1}\right) = \frac{1}{s}$

$\frac{s^3+4s^2+\cancel{s}+4-\cancel{s}}{s^2+1}$

$Y(s) = \frac{1}{s} \cdot \frac{(s^2+1)}{s^3+4s^2+4}$

6. Write the piecewise function $f(t) = \begin{cases} 3t, & 0 \le t < 4 \\ t^2, & t \ge 4 \end{cases}$ in terms of the unit step function. (6 points)

$f(t) = 3t - 3t\, u(t-4) + (t-4)^2 u(t-4)$

7. A series circuit has a capacitor of 10^{-5} F, a resistor of 500 Ω and an inductor of 0.7 H. The initial charge on the capacitor is 10^{-7} C and there is no initial current. Find the charge Q on the capacitor at any time t. What is the **current** of the system at any time t? (10 points)

$C = 10^{-5}$ $R = 500$ $L = .7$

$q(0) = 10^{-7}$
$q'(0) = 0$

$.7q'' + 500q' + 10^5 q = 0$

$r = \dfrac{-500 \pm \sqrt{500^2 - 4(.7)10^5}}{2(.7)} = \dfrac{-500 \pm \sqrt{-3000}}{1.4}$ $-357.14 \pm 123.71\, i$

$q(t) = c_1 e^{-357.14t} \cos(123.71 t) + c_2 e^{-357.14t} \sin(123.71 t)$

$c_1 = 10^{-7}$

$q'(t) = 10^{-7}(-357.14)e^{-357.14t} \cos(123.71t) - 10^{-7}(123.71) \sin(123.71t) + c_2 e^{-357.14t} \sin(123.71t) + 123.71 c_2 e^{-357.14t} \cos(123.71t)$

$\dfrac{-357.14 \times 10^{-7}}{123.71} = c_2 = 2.89 \times 10^{-7}$

Charge

$\boxed{q(t) = e^{-357.14t}(10^{-7} \cos(123.71t) - 2.89 \times 10^{-7} \sin(123.71t))}$

Current

$\boxed{q'(t) = e^{-357.14t}(-7.15 \times 10^{-5} \cos(123.71 t) - 9.08 \times 10^{-5} \sin(123.71 t))}$

8. A mass weighing 8 lbs stretches a spring 16 inches. If the mass is pushed upwards, contracting the spring a distance of 1 inch and then set in motion with a downward velocity of 4 ft/sec, and if there is no damping, and the system is driven by a force of 9cos(2t) pounds, set up the differential equation that models the system. (7 points)

$\dfrac{8\text{lb}}{32} = m = \dfrac{1}{4}$ $8 = k \cdot \dfrac{16}{12} \Rightarrow k = 8 \cdot \dfrac{12}{16} = 6$

$\dfrac{1}{4} y'' + 6y = 9\cos(2t)$ $y(0) = \dfrac{1}{12}$ $y'(0) = -4$

9. Consider the following second order differential equations that model mechanical vibrations. Determine whether the systems they model are undamped, underdamped, critically damped or overdamped. If the system is undamped, state the natural frequency of the system. If the system is underdamped, state the quasi-frequency. (4 points each)

a. $4y'' + y = 0, y(-2) = 1, y'(-2) = -1$

no damping $\Rightarrow$ undamped

$4r^2 + 1 = 0$

$r = \pm \frac{1}{2} i$ natural frequency $\omega = \frac{1}{2}$

b. $9y'' + 12y' + 4y = 0, y(0) = 2, y'(0) = -1$

3 3·2·2 2

repeated roots critically damped.

no frequency

10. Below are solutions to mechanical vibration problems. For each solution state whether the system experiences beats or resonance or neither. Also, state which part of the solution is the transient solution, and which is the steady state solution. (4 points each)

a. $y(t) = \frac{4}{3}\cos(11t) - \frac{7}{9}\sin(11t) + \frac{11}{10}\sin(10t)$

beats. all terms survive in time so no transient all steady state

b. $y(t) = \underbrace{3e^{-\frac{t}{10}}\sin(\sqrt{3}t) - 5e^{-\frac{t}{10}}\cos(\sqrt{3}t)}_{\text{transient}} + \underbrace{\cos(2t) - 12\sin(2t)}_{\text{Steady state}}$

no beats or resonance (may be temporary quasi-beats)

c. $y(t) = \frac{1}{4}\sin(2t) - \frac{1}{8}t\sin(2t) + \frac{1}{6}t\cos(2t)$

resonance all terms survive through time, no transient, all steady state