Math 2255, Exam #1, Spring 2013 Name KEY

Instructions: Show all work. Answers with no work will be graded all or nothing unless the point of the problem is to show the work (in which case, no work will receive no credit). Use exact values (fractions and square roots, etc.) unless the problem tells you to round, is a word problem, or begins with decimal values.

1. For the following differential equations, determine the order of the equation, whether it is linear or non-linear, and whether it is ordinary or partial. (3 points each)
   a. $\frac{y\dot{y} + ty^2 = 0}{y}$ $\dot{y} + ty = 0$ linear, first order, ordinary

   b. $t^2 \frac{d^2y}{dt^2}\frac{dy}{dt} + t\frac{dy}{dt} + 2y = sint$ nonlinear, second order, ordinary

   c. $y''' + ty'' + (cos^2 t)y = t^3$ linear, 3rd order, ordinary

   d. $(x^2 + y)dx + (2 - x)dy = 0$
   $\frac{x^2+y}{2-x} = -y'$ $y' = \frac{-y}{2-x} - \frac{x^2}{2-x}$ linear, first order, ordinary

   e. $u_{xx} + u_{yy} + u_x + u_y + u = 0$
   linear, second order, partial

2. Verify that $y = t^{-2} \ln t$ is a solution to the differential equation $t^2y'' + 5ty' + 4y = 0$. (15 points).

$y = t^{-2}\ln t$ $y' = -2t^{-3}\ln t + t^{-2}\cdot\frac{1}{t} = -2t^{-3}\ln t + t^{-3}$

$y'' = 6t^{-4}\ln t - 2t^{-3}\cdot\frac{1}{t} - 3t^{-4} = 6t^{-4}\ln t - 5t^{-4}$

$t^2(6t^{-4}\ln t - 5t^{-4}) + 5t(-2t^{-3}\ln t + t^{-3}) + 4t^{-2}\ln t =$

$6t^{-2}\ln t - 5t^{-2} - 10t^{-2}\ln t + 5t^{-2} + 4t^{-2}\ln t =$

$t^{-2}\ln t\underbrace{(6-10+4)}_{=0} + t^{-2}\underbrace{(-5+5)}_{=0} = 0$ ✓

it is a solution

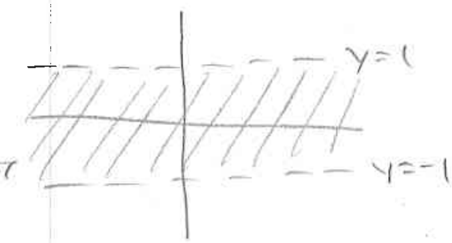
3. Find the region in the t-y plane where the differential equation $y' = t^2(1-y^2)^{1/2}$ would have a unique solution. Sketch the region. (7 points)

$$f(t,y) = t^2(1-y^2)^{1/2}$$

$$1-y^2 \geq 0 \Rightarrow 1 \geq y^2$$

$$\frac{\partial f}{\partial y} = t^2\left(\tfrac{1}{\cancel{2}}\right)(1-y^2)^{-1/2}\cdot(-\cancel{2}y) = \frac{-t^2y}{\sqrt{1-y^2}}$$

$$1-y^2 > 0 \Rightarrow 1 \geq y^2$$



Solutions exist inside here not including boundaries

$y \in (-1, 1)$

no restriction on t

4. A mixing tank initially contains 3000 gallons of brine which contains 25 pounds of salt in solution. A new brine containing 1.5 pounds of salt per gallon begins entering the tank at the rate of 2 gal/minute while the well-stirred mixture leaves the tank at the same rate. Assuming the mixture is kept uniform, find the amount of salt in the tank at the end of an hour. (12 points)

$$R_{in} = \frac{1.5\text{ lbs}}{\text{gal}} \cdot \frac{2\text{ gal}}{\text{min}} = \frac{3\text{ lbs}}{\text{min}} \qquad A(0) = 25\text{ lbs}$$

$$R_{out} = \frac{A}{3000} \cdot \frac{2\text{ gal}}{\text{min}} = \frac{A}{1500}\ \frac{\text{lbs}}{\text{min}}$$

$$\frac{dA}{dt} = 3 - \frac{A}{1500} = \frac{-1}{1500}(A - 4500)$$

$$\int \frac{dA}{A-4500} = \int \frac{-1}{1500}\,dt \rightarrow \ln|A-4500| = \frac{-1}{1500}t + C$$

$$A = 4500 + A_0 e^{\frac{-1}{1500}t} \qquad 25 = 4500 + A_0 e^{0}$$

$$A_0 = -4475$$

$$A(t) = 4500 - 4475e^{\frac{-1}{1500}t} \qquad 1\text{ hour} = 60\text{ min}$$

$$A(60) = 4500 - 4475e^{\frac{-60}{1500}} \approx 200.47\text{ lbs of salt.}$$

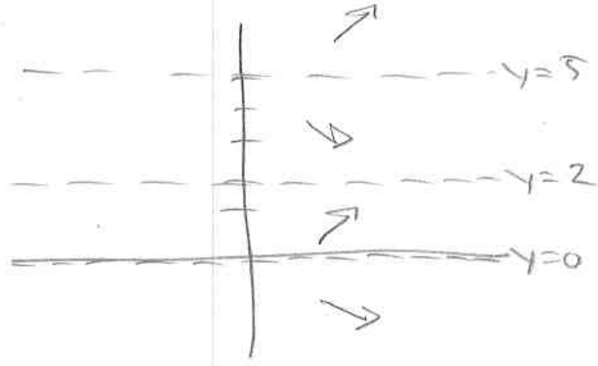
5. Sketch the direction field for the graph $y' = y(y-2)(y-5)$. State whether each equilibrium is stable, unstable or semi-stable, and for each equilibrium greater than zero, state whether it's a threshold or carrying capacity. (10 points)

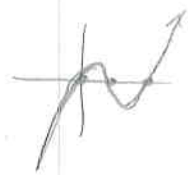
$y' = 0 = y(y-2)(y-5)$

$y=0$ unstable

$y=2$ stable / carrying capacity

$y=5$ unstable / threshold





6. For the following equations, determine the solution method (integrating factor/linear, separable equations, homogeneous (substitution y=vx), Bernoulli equations, exact equation (check for integrating factor if need be) or if the problem must be done numerically. Do not solve the equations. (4 points each)

a. $6xydx + (4y + 9x^2)dy = 0$

$M_y = 6x \quad N_x = 18x$

$\frac{18x - 6x}{6xy} = \frac{12x}{6xy} = \frac{2}{y}$

$\frac{\partial\mu}{\partial y} = \frac{2}{y}\mu \Rightarrow \int \frac{\partial\mu}{\mu} = \int \frac{2}{y}dy \Rightarrow \ln\mu = 2\ln y \Rightarrow \mu = y^2$

$6xy^3 dx + (4y^3 + 9x^2y^2)dy = 0$

$(\mu M)_y = 18xy^2$

$(\mu N)_x = 18xy^2$. EXACT

b. $y' = t + 2y$

$y' - 2y = t$ linear/integrating factor

c. $\frac{yy' + \frac{1}{x}y^2 = x\sqrt{y}}{y} \Rightarrow y' + \frac{1}{x}y = xy^{-1/2}$ bernoulli

d. $xy' = (1-y^2)^{1/2}$ $\quad \frac{dy}{\sqrt{1-y^2}} = \frac{1}{x}dx$ separable

e. $\frac{dy}{dx} = \frac{x}{y} + \frac{y}{x} + 1 = \frac{x^2+y^2+xy}{xy}$ homogeneous

f. $y' = (1-t^2-y^2)^{1/2}$ not linear, not separable, not homogeneous, not bernoulli, not exact $\Rightarrow$ numerical

7. Solve the differential equation $xdx + ye^{-x}dy = 0, y(0) = 1$ (8 points)

$$\frac{ye^{-x}dy}{e^{-x}} = \frac{-xdx}{e^{-x}} \qquad \int y\,dy = \int -xe^{x}dx$$

+/-	u	dv
+	-x	e^x
-	-1	e^x
+	0	e^x

$$\frac{y^2}{2} = -xe^{x} + e^{x} + C$$

$$1 = 0 + 2 + C \Rightarrow C = -1$$

$$y^2 = -2xe^{x} + 2e^{x} - 1$$

$$\boxed{y = \sqrt{2e^{x} - 2xe^{x} - 1}}$$

8. Solve the linear differential equation $x^2y' + xy = 1$ either by integrating factor. Are there any initial conditions that would not produce a unique solution? (15 points)

$$y' + \frac{1}{x}y = \frac{1}{x^2} \qquad \mu = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$$xy' + y = \frac{1}{x}$$

$$\int (xy)' = \int \frac{1}{x}dx \Rightarrow \frac{\ln|x| + C}{x} = \frac{xy}{x}$$

$$\boxed{y = \frac{\ln|x|}{x} + \frac{C}{x}}$$

x may not equal zero ($x \neq 0$) all other initial conditions produce a unique solution

9. Solve the exact equation $(x+y)^2dx + (2xy + x^2 - 1)dy = 0, y(1) = 1$. (12 points)

$$M_y = 2(x+y) = 2x + 2y$$

$$N_x = 2y + 2x \checkmark$$

$$\int (x+y)^2 dx = \int x^2 + 2xy + y^2 dx = \frac{1}{3}x^3 + x^2y + xy^2 + f(y)$$

$$\int 2xy + x^2 - 1\, dy = xy^2 + x^2y - y + g(x)$$

$$\psi(x,y): \frac{1}{3}x^3 + xy^2 + x^2y - y = C$$

$$\frac{1}{3}(1) + (1)(1) + \cancel{(1)(1)} - \cancel{(1)} = C$$

$$4/3 = C$$

$$\boxed{\frac{1}{3}x^3 + x^2y + xy^2 - y = 4/3}$$

10. Use an appropriate substitution to convert the homogeneous differential equation $y' = \frac{x^3+3xy^2}{3y^3}$ into a separable one. You should separate the variables but you do not need to integrate. (8 points)

$$y = vx$$

$$y' = v'x + v$$

$$v'x + v = \frac{x^3 + 3x^3v^2}{3v^3x^3} = \frac{\cancel{x^3}(1+3v^2)}{3v^3\cancel{x^3}}$$

$$v'x \cancel{+v} = \frac{1+3v^2}{3v^3} - v\frac{3v^3}{3v^3} = \frac{1+3v^2-3v^4}{3v^3}$$

$$\int \frac{3v^3}{1+3v^2-3v^4}dv = \int \frac{1}{x}dx$$

Separated

Stop here.

11. Make an appropriate substitution to convert the Bernoulli equation $y' + 2xy = xy^2$ to a linear one. (8 points)

$n=2$

$(1-n)y^{-n} = (1-2)y^{-2} = -1y^{-2}$

$-1y^{-2}y' - 2xy^{-1} = -x$

let $z = y^{-1}$

$z' = -1y^{-2}y'$

$z' - 2xz = -x$

Stop here

12. For the differential equation $y' = 2 - 2t + y^2, y(0) = 1$, numerically approximate the value of y(1) using h=0.2. (9 points)

$x_0 = 0, \quad y_0 = 1$

$m_0 = 2 - 2(0) + (1)^2 = 3$

$y_1 = 3(.2) + 1 = 1.6$

$x_1 = .2 \quad y_1 = 1.6$

$m_1 = 2 - 2(.2) + (1.6)^2 = 4.16$

$y_2 = 4.16(.2) + 1.6 = 2.432$

$x_2 = .4 \quad y_2 = 2.432$

$m_2 = 2 - 2(.4) + (2.432)^2 = 7.114624$

$y_3 = 7.114624(.2) + 2.432 = 3.8549248$

$x_3 = .6 \quad y_3 = 3.8549248$

$m_3 = 2 - 2(.6) + (3.8549)^2 = 15.66044521$

$y_4 = 15.66044521(.2) + 3.8549248 = 6.987013843$

$x_4 = .8 \quad y_4 = 6.987013843$

$m_4 = 2 - 2(.8) + (6.987013843)^2 = 49.21836244$

$y_5 = 49.21836244(.2) + 6.987013843 = 16.83068633$

$x_5 = 1.0 \quad \boxed{y_5 = 16.8307}$