Homework #3 Math 285 Spring 2012

3.3

4. $e^{2-\frac{\pi}{2}i} = e^2 e^{-\frac{\pi}{2}i} = e^2\left(\underbrace{\cancel{\cos(-\tfrac{\pi}{2})}}_{0} + i\sin(-\tfrac{\pi}{2})\right) = ie^2(-1) = -ie^2$

5. $z^{1-i} = e^{(\ln 2)(1-i)} = e^{\ln 2 - i\ln 2} = e^{\ln 2}(\cos\ln 2 - i\sin\ln 2) =$

$(e^{\ln 2}\cos\ln 2) - i(e^2\sin\ln 2)$

11. $y'' + 6y' + 13y = 0 \qquad r^2 + 6r + 13 = 0$

$$\frac{-6 \pm \sqrt{36-52}}{2} = -3 \pm \frac{4i}{2} = -3 \pm 2i$$

$y_1 = e^{-3t}\cos 2t \qquad y_2 = e^{-3t}\sin 2t$

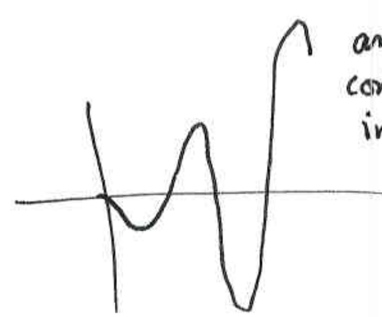
$y(t) = Ae^{-3t}\cos 2t + Be^{-3t}\sin 2t$

19. $y'' - 2y' + 5y = 0 \qquad y(\frac{\pi}{2}) = 0 \qquad y'(\frac{\pi}{2}) = 2$

$r^2 - 2r + 5 = 0$

$$\frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i \qquad y_1 = e^t\cos 2t \qquad y_2 = e^t\sin 2t$$

$y(t) = Ae^t\cos 2t + Be^t\sin 2t$



amplitude continues increasing

$0 = Ae^{\pi/2}\underbrace{\cos\pi}_{-1} + \cancel{Be^{\pi/2}}(0)$

$A = 0$

$y'(t) = 2Be^t\cos 2t + Be^t\sin 2t$

$2 = 2Be^{\pi/2}\underbrace{\cos\pi}_{(-1)} + \cancel{Be^{\pi/2}}(0)$

$2 = -2Be^{\pi/2} \Rightarrow -1 = Be^{\pi/2} \Rightarrow B = -e^{-\pi/2}$

$y(t) = -e^{-\pi/2}\cdot e^t\sin 2t = -e^{(t-\pi/2)}\sin 2t$

27. $W = \begin{vmatrix} e^{\lambda t}\cos\mu t & e^{\lambda t}\sin\mu t \\ \lambda e^{\lambda t}\cos\mu t + e^{\lambda t}\cdot(-\mu\sin\mu t) & \lambda e^{\lambda t}\sin\mu t + e^{\lambda t}\cdot\mu\cos\mu t \end{vmatrix} = \cancel{\lambda e^{2\lambda t}\cos\mu t\sin\mu t} + \mu e^{2\lambda t}\cos^2\mu t - \cancel{\lambda e^{2\lambda t}\cos\mu t\sin\mu t} + \mu e^{2\lambda t}\sin^2\mu t$

$= \mu e^{2\lambda t}(\cos^2\mu t + \sin^2\mu t) =$

$\mu e^{2\lambda t}$

29. $e^{it} = \cos t + i\sin t$

$\sin(-t) = -\sin t$ (odd)
$\cos(-t) = \cos t$ (even)

$e^{-it} = \cos t - i\sin t$

$$\frac{e^{it}+e^{-it}}{2} = \frac{2\cos t}{2} \Rightarrow \cos t = \frac{e^{it}+e^{-it}}{2}$$

$e^{it} = \cos t + i\sin t$

$-e^{-it} = -\cos t + i\sin t$

$$\frac{e^{it}-e^{-it}}{2i} = \frac{2i\sin t}{2i} \Rightarrow \sin t = \frac{e^{it}-e^{-it}}{2i}$$

3.4

8. $16y'' + 24y' + 9y = 0$

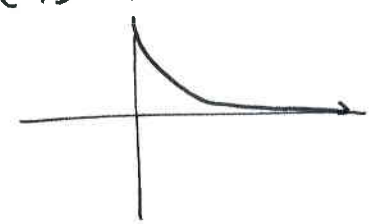
$16r^2 + 24r + 9 = 0$ $\quad (4r+3)(4r+3) = 0$

$r = -3/4$

$y_1 = e^{-3/4t}$ $\quad y_2 = te^{-3/4t}$

$y(t) = Ae^{-3/4t} + Bte^{-3/4t}$

14. $y'' + 4y' + 4y = 0$ $\quad y(-1) = 2 \quad y'(-1) = 1$



$r^2 + 4r + 4 = 0$

$(r+2)^2 = 0$ $\quad r = 2$

$y_1 = e^{-2t}$ $\quad y_2 = te^{-2t}$

$y(t) = Ae^{-2t} + Bte^{-2t}$ $\quad y'(t) = -2Ae^{-2t} + Be^{-2t} + Bte^{-2t}(-2)$

$2 = Ae^2 - Be^2$ $\quad 1 = -2e^2A + Be^2 + 2Be^2$

$2e^{-2} = A - B$ $\quad e^{-2} = -2A + 3B$

$A = 7e^{-2}$

$B = 5e^{-2}$

$y(t) = 7e^{-2}e^{-2t} + 5e^{-2}te^{-2t}$

$= 7e^{-2(t+1)} + 5te^{-2(t+1)}$

24. $t^2 y'' + 2ty - 2y = 0 \qquad t>0 \qquad y_1(t) = t$

$y_2(t) = v(t)\cdot t$

$t^2(v''t+2v') + 2t(v't+v) - 2(vt) = 0 \qquad y_2'(t) = v't + v$

$t^3 v'' + v'(2t^2 + 2t^2) - v\cancel{(2t-2t)} = 0 \qquad y_2''(t) = v''t + v' + v' = v''t + 2v'$

$$\frac{t^3 v'' + 4t^2 v' = 0}{t^2}$$

let $u = v'$

$u' = v''$

$tv'' + 4v' = 0$

$t\cdot \frac{du}{dt} = -4u \Rightarrow \int \frac{du}{u} = \int -\frac{4}{t}\,dt$

$\ln u = -4\ln t = \ln t^{-4}$

$u = t^{-4}$

$\int v' = \int t^{-4}$

$v = \underbrace{-1/3}\, t^{-3}$

we can account for this later in the general solution

$y_2 = \underbrace{t^{-3}}_{v} \cdot t = t^{-2}$

$y(t) = At + B/t^2$

27. $xy'' - y' + 4x^3 y = 0 \qquad x>0 \qquad y_1(x) = \sin x^2$

$y_2(x) = v(x)\sin x^2$

$y_2' = v'\sin x^2 + 2xv\cos x^2$

$y_2'' = v''\sin x^2 + 2xv'\cos x^2 + 2v\cos x^2 + 2xv'\sin x^2 - 4x^2 v\sin x^2$

$x(v''\sin x^2 + 2xv'\cos x^2 + 2v\cos x^2 + 2xv'\sin x^2 - 4x^2 v\sin x^2) - (v'\sin x^2 + 2xv\cos x^2) + 4x^3(v\sin x^2) = 0$

$v''(x\sin x^2) + v'(2x^2\cos x^2 + 2x^2\sin x^2 - \sin x^2) + v(\cancel{2x\cos x^2} - \cancel{4x^3\sin x^2} - \cancel{2x\cos x^2} + \cancel{4x^3\sin x^2}) = 0$

$v''(x\sin x^2) + v'(2x^2 - \sin x^2) + \cancel{v(0)} = 0$

27 contd

$u = v'$

$u' = v''$

$$x \sin x^2 \frac{du}{dx} = (\sin x^2 - 2x^2)u$$

$$\frac{du}{u} = \frac{\sin x^2 - 2x^2}{x \sin x^2} dx$$

$$\int \frac{du}{u} = \frac{\cancel{\sin x^2}}{x \cancel{\sin x^2}} - \frac{2x^{\cancel{2}}}{\cancel{x}\sin x^2} = \int \frac{1}{x} - 2x \csc x^2 \, dx$$

$w = x^2$

$du = 2x$

$\int \csc w \, dw$

$$\ln u = \ln x + \ln|\csc x^2 + \cot x^2|$$

$$\ln u = \ln[x(\csc x^2 + \cot x^2)]$$

$$u = x(\csc x^2 + \cot x^2) = v'$$

$$v = \int x \csc x^2 + x \cot x^2 \, dx$$

$$= -\frac{1}{2} \ln|\csc x^2 + \cot x^2| + \frac{1}{2}\ln|\sin x^2|$$

$$= \ln \sqrt{\frac{\sin x^2}{\csc x^2 + \cot x^2} \cdot \frac{\sin x^2}{\sin x^2}} = \ln \sqrt{\frac{\sin^2 x^2}{1 + \cos x^2}} =$$

$$\ln\left(\frac{\sin x^2}{\sqrt{1+\cos x^2}}\right)$$

$$y_2(x) = \sin x^2 \ln\left(\frac{\sin x^2}{\sqrt{1+\cos x^2}}\right)$$

3.5

#5. $y'' + 9y = t^2 e^{3t} + 6$

$r^2 + 9 = 0$

$r = \pm 3i$

$y_1 = \cos 3t \quad y_2 = \sin 3t$

$Y(t) = (At^2 + Bt + C)e^{3t} + D$

$Y'(t) = (2At + B)e^{3t} + 3(At^2 + Bt + C)e^{3t}$

$Y''(t) = 2Ae^{3t} + 3(2At + B)e^{3t} +$

$3(At^2 + Bt + C) * 3e^{3t} + 3(2At + B)e^{3t}$

5 contd

$Y'' + 9y = t^2 e^{3t} + 6$

$2Ae^{3t} + (6At + 3B)e^{3t} + (3At^2 + 3Bt + 3C)e^{3t} + (6At + 3B)e^{3t}$

$+ 9(At^2 + Bt + C)e^{3t} + 9D$

$9D = 6 \quad D = 6/9 = 2/3$

$3At^2e^{3t} + 9At^2e^{3t} = t^2e^{3t} \Rightarrow 3A + 9A = 1$

$12A = 1 \Rightarrow A = 1/12$

$6Ate^{3t} + 3Bte^{3t} + 6Ate^{3t} + 9Bte^{3t} = 0$

$12A + 12B = 0$

$1 + 12B = 0 \Rightarrow 12B = -1 \Rightarrow B = -1/12$

$2Ae^{3t} + 3Be^{3t} + 3Ce^{3t} + 3Be^{3t} + 9Ce^{3t} = 0$

$2A + 6B + 12C = 0 \Rightarrow 1/6 + 1/2 + 12C = 0$

$12C = -2/3 \quad C = -1/18$

$Y(t) = (1/12\,t^2 - 1/12\,t - 1/18)e^{3t} + 2/3$

$y(t) = E\cos 3t + F\sin 3t + (1/12\,t^2 - 1/12\,t - 1/18)e^{3t} + 2/3$

8. $y'' + y = 3\sin 2t + t\cos 2t$

$r^2 + 1 = 0$

$r = \pm i \qquad y_1 = \cos t,\ y_2 = \sin t$

$Y(t) = A\sin 2t + B\cos 2t + Ct\sin 2t + Dt\cos 2t$

$Y'(t) = A2\cos 2t - 2B\sin 2t + C\sin 2t + 2Ct\cos 2t + D\cos 2t - 2Dt\sin 2t$

$Y''(t) = -4A\sin 2t - 4B\cos 2t + 2C\cos 2t + 2C\cos 2t - 4Ct\sin 2t -$

$2D\sin 2t - 4Dt\cos 2t - 2D\sin 2t$

8 contd

$\sin 2t(-4A - 2D - 2D + A) = 3\sin 2t \quad \Rightarrow -3A - 4D = 3$

$\cos 2t(-4B + \cancel{2C} + \cancel{2C} + B) = 0 \quad \Rightarrow B = 0$

$t\sin 2t(-4C + C) = 0 \quad \Rightarrow C = 0$

$t\cos 2t(-4D + D) = t\cos 2t \quad \Rightarrow -3D = 1 \quad D = -1/3$

$-3A - (-1/3)4 = 3$

$-3A + 4/3 = 3$

$-3A = 5/3 \quad \Rightarrow \quad A = -5/9$

$Y(t) = -5/9 \sin 2t - 1/3\, t\cos 2t$

$y(t) = E\cos t + F\sin t - 5/9 \sin 2t - 1/3\, t\cos 2t$

12. $y'' - y' - 2y = \cosh 2t = \frac{1}{2}e^{-2t} + \frac{1}{2}e^{2t}$

$r^2 - r - 2 = 0$

$(r-2)(r+1) = 0 \qquad y_1 = e^{2t},\ y_2 = e^{-t}$

$r = 2, -1$

$Y(t) = Ate^{2t} + Be^{-2t}$

$Y'(t) = Ae^{2t} + 2Ate^{2t} + -2Be^{-2t}$

$Y''(t) = \underbrace{2Ae^{2t} + 2Ae^{2t}}_{4Ae^{2t}} + 4Ate^{2t} + 4Be^{-2t}$

$4Ae^{2t} + 4Ate^{2t} + 4Be^{-2t} - Ae^{2t} - 2Ate^{2t} + 2Be^{-2t} - 2Ate^{2t} - Be^{-2t}$

$te^{2t}(4A - 2A - 2A) = 0$

$e^{2t}(4A - A) = \frac{1}{2} \quad \Rightarrow 3A = 1/2 \quad A = 1/6$

$e^{-2t}(4B + 2B - B) = 1/2 \quad 5B = 1/2 \Rightarrow B = 1/10$

$Y(t) = \frac{1}{6}te^{2t} + \frac{1}{10}e^{-2t} \quad \Rightarrow y(t) = Ce^{2t} + De^{-t} + \frac{1}{6}te^{2t} + \frac{1}{10}e^{-2t}$

6. $y'' - 2y' - 3y = 3te^{2t}$ $\qquad y(0)=1,\ y'(0)=0$

$r^2 - 2r - 3 = 0$

$(r-3)(r+1) = 0$ $\qquad y_1 = e^{3t},\ y_2 = e^{-t}$

$r = 3, r = -1$ $\qquad Y(t) = (At+B)e^{2t}$

$Y'(t) = Ae^{2t} + 2(At+B)e^{2t}$

$Y''(t) = \underbrace{2Ae^{2t} + 2Ae^{2t}}_{4Ae^{2t}} + 4(At+B)e^{2t}$

$4Ae^{2t} + \underline{4Ate^{2t}} + 4Be^{2t} - 2Ae^{2t} - \underline{4Ate^{2t}} - 4Be^{2t} - \underline{3Ate^{2t}} - 3Be^{2t}$

$te^{2t}(\cancel{4A} - \cancel{4A} - 3A) = 3te^{2t} \Rightarrow -3A = 3 \quad A = -1$

$e^{2t}(4A + \cancel{4B} - 2A - \cancel{4B} - 3B) = 0 \Rightarrow -B = 0$

$Y(t) = -te^{2t}$

$y(t) = Ce^{3t} + De^{-t} - te^{2t}$

$1 = C + D$ $\qquad D = 1 - C$

$y'(t) = 3Ce^{3t} - De^{-t} - e^{2t} - 2te^{2t}$

$0 = 3C - D - 1$

$1 = 3C - (1-C) \Rightarrow 1 = 3C - 1 + C \Rightarrow 2 = 4C \Rightarrow C = 1/2$

$D = 1 - 1/2 = 1/2$

$y(t) = \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} - te^{2t}$

18. $y'' + 2y' + 5y = 4e^{-t}\cos 2t$ $\qquad y(0)=1,\ y'(0)=0$

$r^2 + 2r + 5 = 0$

$\frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$ $\qquad y_1 = e^{-t}\cos 2t \qquad y_2 = e^{-t}\sin 2t$

$Y(t) = Ate^{-t}\cos 2t + Bte^{-t}\sin 2t$

$Y'(t) = Ae^{-t}\cos 2t - Ate^{-t}\cos 2t - 2Ate^{-t}\sin 2t + Be^{-t}\sin 2t - Bte^{-t}\sin 2t + 2Bte^{-t}\cos 2t$

18 contd

$Y'(t) = e^{-t}\cos 2t(-At + A + 2Bt) + e^{-t}\sin 2t(-Bt + B - 2At)$

$Y''(t) = e^{-t}\cos 2t(At - 2A - 2Bt + 2B - 2Bt + 2B - 4At)$

$\quad + e^{-t}\sin 2t(2At - 2A + Bt - 2B + 2At - 2A - 4Bt)$

$te^{-t}\sin 2t(2A + B + 2A - 4B - 2B - 4A + 5B) = 0 \Rightarrow 0 = 0$

$e^{-t}\sin 2t(-2A - 2B - 2A + 2B) = 0 \Rightarrow A = 0$

$te^{-t}\cos 2t(A - 2B - 2B - 4A - 2A + 4B + 5A) = 0 \Rightarrow 0 = 0$

$e^{-t}\cos 2t(-2A + 2B + 2B + 2A) = 4 \Rightarrow 4B = 4 \Rightarrow B = 1$

$Y(t) = te^{-t}\sin 2t$

$y(t) = Ce^{-t}\cos 2t + De^{-t}\sin 2t + te^{-t}\sin 2t$

$1 = C$

$y'(t) = -Ce^{-t}\cos 2t - 2Ce^{-t}\sin 2t - De^{-t}\sin 2t + 2De^{-t}\cos 2t +$
$\quad e^{-t}\sin 2t - te^{-t}\sin 2t + 2te^{-t}\cos 2t$

$0 = -C + 2D \Rightarrow 2D = 1 \Rightarrow D = \frac{1}{2}$

$y(t) = e^{t}\cos 2t + \frac{1}{2}e^{-t}\sin 2t + te^{-t}\sin 2t$

3.6.

3. $y'' + 2y' + y = 3e^{-t}$

$r^2 + 2r + 1 = 0$

$r = -1$

$y_1 = e^{-t} \quad y_2 = te^{-t}$

$W = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & e^{-t} - te^{-t} \end{vmatrix} = e^{-2t} - \cancel{te^{-2t}} + \cancel{te^{-2t}} = e^{-2t}$

3 contd

$$Y(t) = -e^{-t}\int \frac{t\cancel{e^{t}}\cdot 3\cancel{e^{t}}}{\cancel{e^{2t}}}\,dt + te^{-t}\int \frac{\cancel{e^{t}}\cdot 3\cancel{e^{t}}}{\cancel{e^{2t}}}\,dt$$

$$= -e^{-t}\int 3t\,dt + te^{-t}\int 3\,dt$$

$$-e^{-t}\tfrac{3}{2}t^2 + te^{-t}3t = \tfrac{3}{2}t^2e^{-t}$$

9. $4y'' + y = 2\sec(t/2)$ $\quad -\pi < t < \pi$

$4r^2 + 1 = 0$

$r = \pm\frac{1}{2}i$ $\quad y_1 = \cos(t/2)$ $\quad y_2 = \sin(t/2)$

$$W = \begin{vmatrix} \cos t/2 & \sin t/2 \\ -\frac{1}{2}\sin t/2 & 1/2\cos t/2 \end{vmatrix} = \frac{1}{2}\cos^2 t/2 + 1/2\sin^2 t/2 = 1/2$$

$$Y(t) = -\cos(t/2)\int \frac{\sin(t/2)\cdot 2\sec(t/2)}{1/2}\,dt + \sin(t/2)\int \frac{\cancel{\cos(t/2)}\,2\cancel{\sec(t/2)}}{1/2}\,dt$$

$$-\cos(t/2)\int 4\tan(t/2)\,dt + \sin(t/2)\int 4\,dt$$

$$+\cos(t/2)\cdot 8\ln|\cos t/2| + 4\sin t/2\cdot t$$

$$y(t) = A\cos(t/2) + B\sin(t/2) + 8\cos(t/2)\ln|\cos t/2| + 4t\sin(t/2)$$

14. $t^2y'' - t(t+2)y' + (t+2)y = 2t^3$ $\quad t > 0$, $\quad y_1(t) = t$

$y_2(t) = te^t$

$y_1' = 1$ $\quad \Rightarrow 0 - t(t+2)1 + (t+2)t) = 0$ ✓

$y_1'' = 0$

$y_2' = e^t + te^t$

$y_2'' = e^t + e^t + te^t = 2e^t + te^t$

$2t^2e^t + t^3e^t - t(t+2)(e^t + te^t) + (t+2)(te^t) = 0$ ✓

14 contd.

$$W = \begin{vmatrix} t & te^t \\ 1 & e^t + te^t \end{vmatrix} = \cancel{te^t} + t^2e^t - \cancel{te^t} = t^2e^t$$

$$Y(t) = -t\int \frac{t\cancel{e^t}\,2t^3}{t^2\cancel{e^t}}\,dt + te^t\int \frac{t\cdot 2t^3}{t^2e^t}\,dt$$

$$= -t\int 2t^2\,dt + te^t\int 2t^2e^{-t}\,dt$$

$$= -t\left[\tfrac{2}{3}t^3\right] + t\cancel{e^t}\left[-2t^2 - 4t - 4\right]\cancel{e^{-t}}$$

$$= -\tfrac{2}{3}t^4 - 2t^3 - 4t^2 - 4t$$

$$y(t) = At + Bte^t - \tfrac{2}{3}t^4 - 2t^3 - 4t^2 - \cancel{4t} \leftarrow \text{combines w/ } At$$

15. $ty'' - (1+t)y' + y = t^2e^{2t} \qquad t>0 \qquad y_1 = 1+t \qquad y_2 = e^t$

$y_1' = 1 \qquad 0 - (1+t)1 + 1+t = 0 \checkmark$

$y_1'' = 0$

$y_2' = e^t \qquad y_2'' = e^t \qquad te^t - (1+t)e^t + e^t = 0 \checkmark$

$$W = \begin{vmatrix} 1+t & e^t \\ 1 & e^t \end{vmatrix} = \cancel{e^t} + te^t - \cancel{e^t} = te^t$$

$$Y(t) = -(1+t)\int \frac{\cancel{e^t}\cdot t^2e^{2t}}{t\cancel{e^t}}\,dt + e^t\int \frac{(1+t)\,t^2\cancel{e^{2t}}^{\,e^t}}{t\cancel{e^t}}\,dt$$

$$= (1+t)\int te^{2t}\,dt + e^t\int (t^2+t)e^t\,dt$$

$$(1+t)\left[\frac{te^{2t}}{2} - \frac{e^{2t}}{4}\right] + e^t\left[(2-2t)e^t + t^2e^t + te^t - e^t\right]$$

$$e^{2t}\left(t^2 - t + 1 + \tfrac{1}{4}(2t^2 + t - 1)\right) = e^{2t}\left(t^2 - t + 1 + \tfrac{1}{2}t^2 + \tfrac{1}{4}t - \tfrac{1}{4}\right)$$

15 contd

$= e^{2t}\left(\tfrac{3}{2}t^2 - \tfrac{3}{4}t + \tfrac{3}{4}\right)$

$y(t) = A(1+t) + Be^{t} + e^{2t}\left(\tfrac{3}{2}t^2 - \tfrac{3}{4}t + \tfrac{3}{4}\right)$

29. $t^2 y'' - 2t y' + 2y = 4t^2 \qquad t>0, \quad y_1(t) = t$

$y_2 = vt$

$y_2' = v't + v$

$y_2'' = v''t + 2v'$

$t^2(v''t + 2v') - 2t(v't + v) + 2(vt) = 0$

$v''(t^3) + v'(\cancel{2t^2 - 2t^2}) + v(\cancel{-2t + 2t}) = 0$

$v''t^3 = 0$

$v'' = 0 \quad \Rightarrow \quad V = (At + B)$

$y_2 = vt = (At+B)t = At^2 + Bt \Leftarrow$ Bt is a multiple of y_1 so we can ignore it

$y_2 = At^2$

$Y(t) = -t\int \frac{\cancel{t^2}\, 4t^2}{\cancel{t^2}}\,dt + t^2 \int \frac{t\, 4\cancel{t^2}}{\cancel{t^2}}\,dt$

$W = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = 2t^2 - t^2 = t^2$

$-t\int 4t^2\,dt + t^2\int 4t\,dt =$

$-t\left(\tfrac{4}{3}t^3\right) + t^2\left(\cancel{\tfrac{4}{2}}^{\,2}t^2\right) = -\tfrac{4}{3}t^4 + 2t^4 = \tfrac{2}{3}t^4$

$y(t) = At^2 + Bt + \tfrac{2}{3}t^4$