Homework #2 Key Math 285 Spring 2012

2.4 #3 $y' + (\tan t)y = \sin t$

$y' = \sin t - (\tan t)y = f(t,y)$

$f(t,y)$ not continuous at $(2k+1)\cdot \frac{\pi}{2} = t \quad k \in \mathbb{Z}$

$\frac{\partial f}{\partial y} = \cos t - (\tan t)$ same points of discontinuity

$y(\pi) = 0 \Rightarrow \boxed{(\frac{\pi}{2}, \frac{3\pi}{2})}$

5. $(4-t^2)y' + 2ty = 3t^2$

$y' = \frac{-2ty}{4-t^2} + \frac{3t^2}{4-t^2} = f(t,y)$

not defined when $4-t^2=0 \Rightarrow$
$t \neq 2, -2$

$\frac{\partial f}{\partial y} = \frac{-2t}{4-t^2}$ no new conditions

for $y(1) = -3$ interval is $\boxed{(-2,2)}$

11. $\frac{dy}{dt} = \frac{1+t^2}{3y-y^2} = f(t,y)$

$3y - y^2 \neq 0$

$y(y-3) \neq 0 \quad \boxed{y \neq 0,\ y \neq 3}$

$\frac{\partial f}{\partial y} = (1+t^2)(-1)(3y-y^2)^{-2}(3-2y)$ no new conditions

13. $y' = -4t/y \longrightarrow y \neq 0$

$\int y\,dy = \int -4t\,dt$

$\frac{1}{2}y^2 = -2t^2 + C \Rightarrow y^2 = -4t^2 + C$

$y(0) = y_0$

$C = y_0^2 \quad \boxed{y^2 = -4t^2 + y_0^2}$

13. cont'd

$$y^2 = -4t^2 + y_0^2$$

$$\Rightarrow y = \pm\sqrt{-4t^2+y_0^2} \qquad y_0^2 > 4t^2$$ or there is no solution (no real solution at least)

28. $t^2y' + 2ty - y^3 = 0$

$$\frac{t^2y'}{t^2} + \frac{2ty}{t^2} = \frac{y^3}{t^2} \rightarrow y' + \frac{2}{t}y = t^{-2}y^3 \qquad n=3$$

$$(* \ y^{-3})(1-3) = -2y^{-3} \qquad -2y^{-3}y' + \frac{2}{t}(-2y^{-2}) = -2t^{-2}$$

$$z = y^{-2}$$

$$z' = -2y^{-3}y'$$

$$z' + \frac{-4}{t}z = -2t^{-2}$$

$$\mu = e^{\int -\frac{4}{t}dt} = e^{-4\ln t} = e^{\ln\frac{1}{t^4}} = \frac{1}{t^4}$$

$$\frac{1}{t^4}z' + -4t^{-5}z = -2t^{-6}$$

$$\int (t^{-4}z)' = \int -2t^{-6}\,dt$$

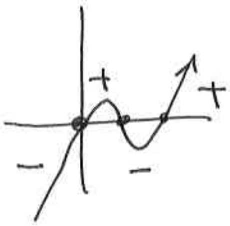
$$t^{-4}z = -\frac{2}{5}t^{-5} + C \qquad * t^4$$

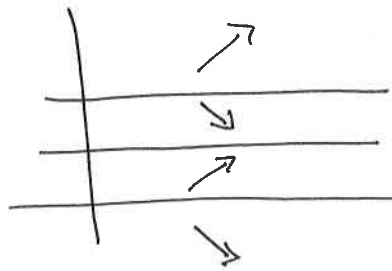
$$z = -\frac{2}{5}t^{-1} + Ct^4$$

$$y^{-2} = -\frac{2}{5}t^{-1} + Ct^4 = \frac{-2}{5t} + Ct^4 = \frac{-2+Ct^5}{5t}$$

$$y^2 = \frac{5t}{Ct^5-2} \Rightarrow y = \pm\sqrt{\frac{5t}{Ct^5-2}}$$

2.5 #3 $\frac{dy}{dt} = y(y-1)(y-2) = 0$ $y=0,\ y=1,\ y=2$ equilibria



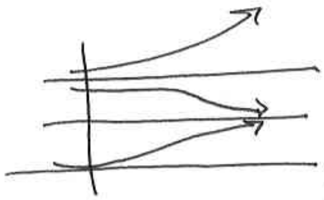


$y=2$ threshold $\Rightarrow$ unstable

$y=1$ carrying capacity = stable

$y=0$

$y_0 > 0$



$y_0=0$

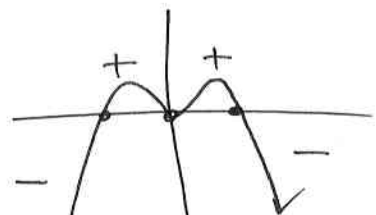
between $y_0 = 0$ & $y_0 = 2$
graph $\to$ $y=1$

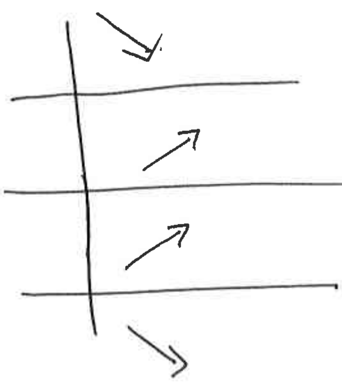
above $y_0 = 2$ graph $\to \infty$

12. $\frac{dy}{dt} = y^2(4-y^2)$ $-\infty < y_0 < \infty$

$y=0 \quad y=\pm 2$

$-y^4$

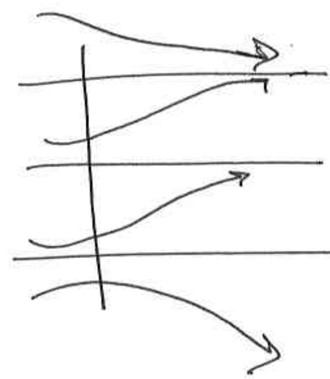




$y=2$ stable

$y=0$ semistable

$y=-2$ unstable



$y=2$

$y=0$

$y=-2$

15. $\frac{dy}{dt} = ry\left[1-\frac{y}{K}\right]$ $y_0 = K/3$

$$y = \frac{y_0 K}{y_0 + (K-y_0)e^{-rt}} = \frac{K^2/3}{K/3 + 2/3K\, e^{-rt}} \cdot \frac{3/K}{3/K} = \frac{K}{1+2e^{-rt}}$$

$$y(\tau) = \frac{2K}{3} = \frac{K}{1+2e^{-rt}} \Rightarrow \frac{3}{2} = 1+2e^{-rt}$$

$$\frac{1}{2} = 2e^{-rt}$$

$$\frac{1}{4} = e^{-rt} \qquad \ln\frac{1}{4} = -rt \qquad \tau = \frac{\ln(1/4)}{-r} \qquad r = .025$$

$$\tau = \frac{\ln(1/4)}{-.025} = 55.45 \text{ years.}$$

b) $y_0/K = \alpha$ $\qquad y(T) = \beta$

$$y_0 = \alpha K$$

$$y = \frac{\alpha K^2}{\alpha K + (1-\alpha)K e^{-rt}} \quad \frac{\frac{1}{K}}{\frac{1}{K}} = \frac{\alpha K}{\alpha + (1-\alpha)e^{-rt}} = \beta K$$

$$\frac{1}{\beta K} = \frac{\alpha + (1-\alpha)e^{-rt}}{\alpha K} \Rightarrow \frac{\alpha K}{\beta K} = \alpha + (1-\alpha)e^{-rt}$$

$$\frac{\alpha}{\beta} - \alpha = \frac{\alpha(1-\beta)}{\beta} = (1-\alpha)e^{-rt} \Rightarrow \frac{\alpha(1-\beta)}{\beta(1-\alpha)} = e^{-rt}$$

$$\ln\left[\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right] = -rT \Rightarrow T = -\frac{1}{r}\ln\left[\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right]$$

$$T = -\frac{1}{.025}\ln\left[\frac{.1(1-.9)}{.9(1-.1)}\right] = -\frac{1}{.025}\ln\left[\frac{.1^2}{.9^2}\right] = 175.78 \text{ years}$$

25. $\frac{dy}{dt} = a - y^2$ $\qquad$ Case #1 if $a < 0$ let $a = -b$

then $\frac{dy}{dt} = -b - y^2 = -(b + y^2) = 0$

$b + y^2 \neq 0$ for $b > 0$ i.e. $a < 0$.

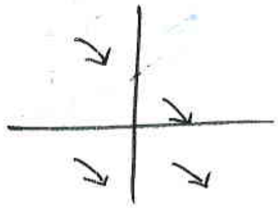
no critical points

Case #2 for $a = 0$ $\qquad \frac{dy}{dt} = y^2$ $\qquad y = 0$ is semistable

case #3 for $a > 0$ $\qquad \frac{dy}{dt} = a - y^2 = (\sqrt{a} - y)(\sqrt{a} + y)$ 2 critical points one stable, one unstable

25 cont'd.

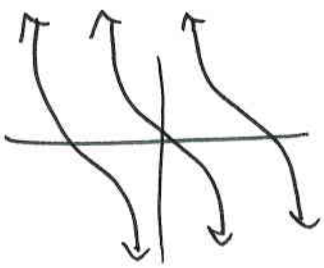
Case #1



slopes everywhere negative
no equilibria

$$\frac{dy}{dt} = -(b+y^2)$$

$$\frac{dy}{b+y^2} = -\,dt$$



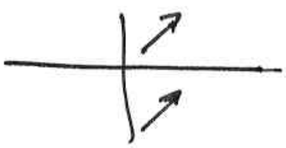
$$\frac{1}{\sqrt{b}}\arctan\frac{y}{\sqrt{b}} = -t + C$$

$$\arctan\frac{y}{\sqrt{b}} = -\sqrt{b}\,t + C$$

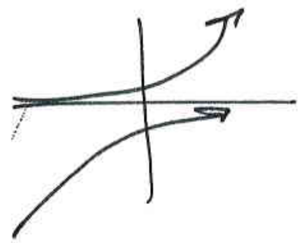
$$\frac{y}{\sqrt{b}} = \tan(-\sqrt{b}\,t + c)$$

$$y = -\sqrt{b}\,\tan(\sqrt{b}\,t + c)$$

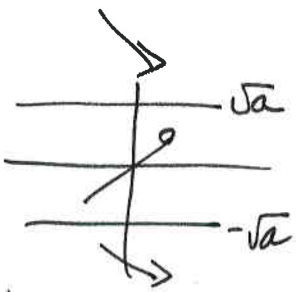
Case # 2

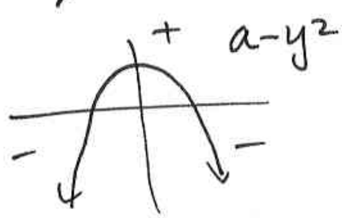


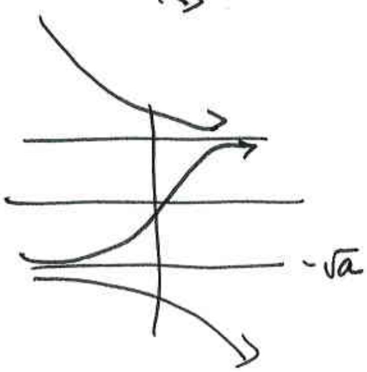
$\rightarrow$



Case #3

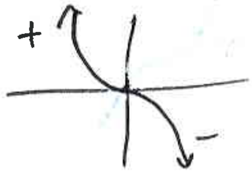






26. $\frac{dy}{dt} = ay - y^3 = y(a-y^2)$

case#1 $a<0$ let $a=-b$ $\Rightarrow -y(b+y^2) = \frac{dy}{dt}$



y=0 unstable

$$\frac{dy}{dt} = -y(b+y^2) \Rightarrow \frac{dy}{y(b+y^2)} = -dt$$

$$\frac{A}{y} + \frac{By+C}{b+y^2} = \frac{1}{y(b+y^2)} = A(b+y^2) + By^2 + Cy$$
$$Ab + Ay^2 + By^2 + Cy$$

$Ab=1$ $\quad C=0 \quad Ay^2+By^2=0$

$A=\frac{1}{b} \quad A+B=0 \quad A=-B \quad B=-\frac{1}{b}$

$$\frac{1}{b}\int \frac{1}{y}\,dy - \frac{1}{b}\int \frac{y}{b+y^2}\,dy = \int -dt$$

$$\frac{1}{b}\ln y - \frac{1}{2b}\ln|b+y^2| = -t+c$$

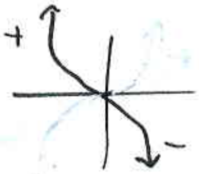
$$\ln y^{1/b} - \ln(b+y^2)^{1/2b} = -t+c$$

$$\ln\left[\frac{y}{\sqrt{b+y^2}}\right]^{1/b} = -t+c$$

$$\left[\frac{y}{\sqrt{b+y^2}}\right]^{1/b} = Ae^{-t} \Rightarrow \frac{y}{\sqrt{b+y^2}} = Ae^{-bt}$$

Case #2 $\quad a=0$

$\frac{dy}{dt} = -y^3$



y=0

$$\int \frac{dy}{y^3} = \int -dt$$

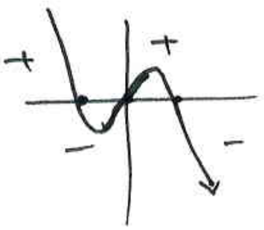
$$-\frac{1}{2y^2} = -t+c$$

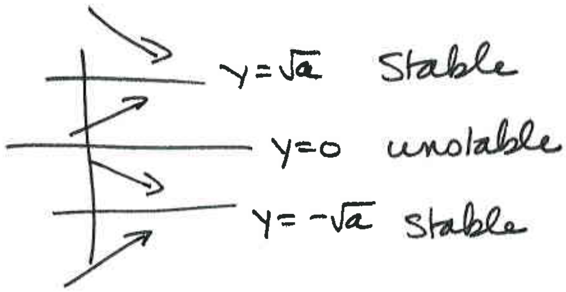
$$\frac{1}{y^2} = 2t+c \Rightarrow y = \pm\frac{1}{\sqrt{2t+c}}$$

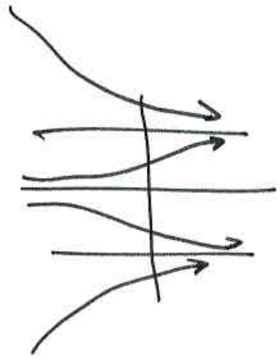
26 cont'd

Case #3 $a>0$

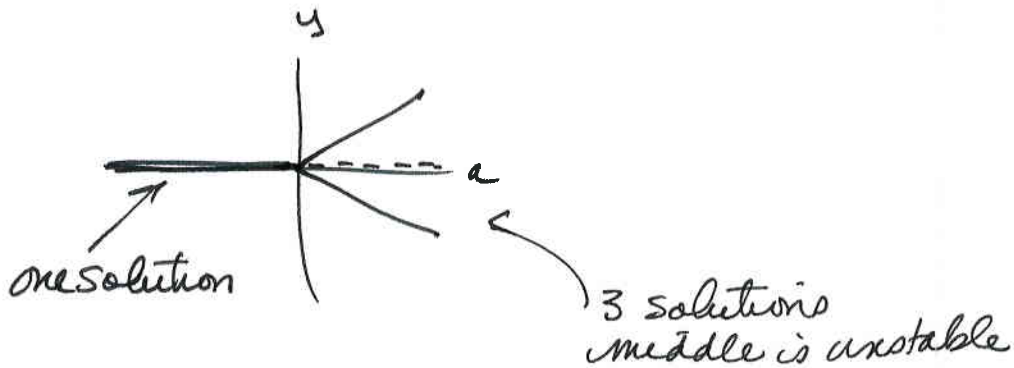
$$\frac{dy}{dt} = y(\sqrt{a}-y)(\sqrt{a}+y)=0$$







$$0 = y(a-y^2)$$



2.6 #2

$$\underset{M}{(2x+4y)} + \underset{N}{(2x-2y)}\, y' = 0$$

$M_y = 4 \neq N_x = 2$ no ψ, not exact

11. $\underset{M}{(x\ln y + xy)}\,dx + \underset{N}{(y\ln x + xy)}\,dy = 0$

$M_y = \frac{x}{y} + x \neq N_x = \frac{y}{x} + y$ no ψ, not exact

21. $y\,dx + (2x - ye^y)\,dy = 0$ $\mu(x,y) = y$

$$\underset{\mu M}{y^2\,dx} + \underset{\mu N}{(2xy - y^2e^y)}\,dy = 0$$

$$(\mu M)_y = 2y = (\mu N)_x = 2x - 0$$

$$\int y^2\,dx = y^2x + f(y) \qquad \int 2xy - y^2e^y\,dy =$$

$\int 2xy - y^2 e^y \, dy$ $\qquad u = y^2 \quad dv = e^y dy$

$xy^2 - [y^2 e^y - \int 2y e^y dy] =$ $\qquad du = 2y\,dy \quad v = e^y$

$xy^2 - y^2 e^y + \int 2y e^y dy$ $\qquad u = y \quad dv = e^y dy$

$xy^2 - y^2 e^y + 2ye^y - \int 2e^y dy$ $\qquad du = dy \quad v = e^y$

$xy^2 - y^2 e^y + 2ye^y - 2e^y + g(x)$

$\psi(x,y) = xy^2 - y^2 e^y + 2ye^y - 2e^y + k$

27. $dx + (x/y - \sin y)dy = 0$

$M = 1 \qquad N$

$M_y = 0 \neq N_x = \frac{1}{y}$

$$\frac{d\mu}{dy} = \frac{1/y - 0}{1}\mu = \frac{1}{y}\mu$$

$$\frac{d\mu}{\mu} = \frac{1}{y}dy \Rightarrow \ln\mu = \ln y \Rightarrow \mu = y$$

$y\,dx + (x - y\sin y)\,dy = 0$

$\mu M \qquad \mu N$

$(\mu M)_y = 1 = (\mu N)_x = 1$

$\int y\,dx = xy + f(y)$ $\qquad u = y \quad dv = \sin y\,dy$

$\int x - y\sin y\,dy = xy - [-y\cos y + \int +\cos y\,dy] =$ $\qquad du = dy \quad v = -\cos y$

$xy + y\cos y - \sin y + g(x)$

$\psi(x,y) = xy + y\cos y - \sin y + k$

3.1

\#5 $y'' + 5y' = 0$

assume $y = e^{rt}$

$y' = re^{rt} \quad y'' = r^2 e^{rt}$

$r^2 e^{rt} + 5re^{rt} = 0$

$e^{rt}(r^2 + 5r) = 0 \quad r(r+5) = 0$

$r = 0 \quad r = -5$

$y_1 = e^{0t} = 1$

$y_2 = e^{-5t}$

$$\boxed{y(t) = A + Be^{-5t}}$$

\#7. $y'' - 9y' + 9y = 0 \implies r^2 - 9r + 9 = 0$

$$r = \frac{9 \pm \sqrt{81-36}}{2} = \frac{9 \pm \sqrt{45}}{2} = \frac{9 \pm 3\sqrt{5}}{2}$$

$y_1 = e^{\frac{9+3\sqrt{5}}{2}t} \quad y_2 = e^{\frac{9-3\sqrt{5}}{2}t}$

$$\boxed{y(t) = Ae^{\frac{9+3\sqrt{5}}{2}t} + Be^{\frac{9-3\sqrt{5}}{2}t}}$$

14. $2y'' + y' - 4y = 0 \quad y(0) = 0, \ y'(0) = 1$

$2r^2 + r - 4 = 0 \quad r = \frac{-1 \pm \sqrt{1^2 + 4(2)(4)}}{2} = \frac{-1 \pm \sqrt{33}}{2}$

$y_1 = e^{\frac{-1+\sqrt{33}}{2}t} \quad y_2 = e^{\frac{-1-\sqrt{33}}{2}t}$

$y(t) = Ae^{\frac{-1+\sqrt{33}}{2}t} + Be^{\frac{-1-\sqrt{33}}{2}t}$

$y'(t) = A\left(\frac{-1+\sqrt{33}}{2}\right)e^{\frac{-1+\sqrt{33}}{2}t} + B\left(\frac{-1-\sqrt{33}}{2}\right)e^{\frac{-1-\sqrt{33}}{2}t}$

$0 = A + B$

$A = -B$

$A = \frac{1}{\sqrt{33}}$

$1 = A\left(\frac{-1+\sqrt{33}}{2}\right) + B\left(\frac{-1-\sqrt{33}}{2}\right)$

$1 = -B\left(\frac{-1+\sqrt{33}}{2}\right) + B\left(\frac{-1-\sqrt{33}}{2}\right)$

$2 = -B(-1+\sqrt{33}) + B(-1-\sqrt{33})$

$2 = \cancel{B} - \sqrt{33}B - \cancel{B} - \sqrt{33}B$

$\frac{2}{-2\sqrt{33}} = \frac{-2\sqrt{33}B}{-2\sqrt{33}}$

$B = -\frac{1}{\sqrt{33}}$

$$\boxed{y(t) = \frac{1}{\sqrt{33}}e^{\frac{-1+\sqrt{33}}{2}} - \frac{1}{\sqrt{33}}e^{\frac{-1-\sqrt{33}}{2}t}}$$

17. $y = c_1 e^{2t} + c_2 e^{-3t}$ (10)

$\Rightarrow r = 2 \quad r = -3 \Rightarrow r - 2 = 0 \quad r + 3 = 0 \Rightarrow$

$(r-2)(r-3) = 0 \Rightarrow r^2 - 5r + 6 = 0$

$\frac{dy}{dt} = y'' - 5y' + 6y$

28. $ay'' + by' + cy = 0 \quad a > 0$

a) real $\Rightarrow b^2 - 4ac \geq 0$

different $\Rightarrow b^2 - 4ac \neq 0$

negative $\Rightarrow -b \pm \sqrt{b^2 - 4ac} < 0$

$-b < \mp \sqrt{b^2 - 4ac}$

i.e. $b > 0$ & $b > \sqrt{b^2 - 4ac}$

b) real $\Rightarrow b^2 - 4ac > 0$

opposite signs $\Rightarrow b > 0$ & $b < \sqrt{b^2 - 4ac}$

or $b < 0$ & $b > \sqrt{b^2 - 4ac}$

c) real $\Rightarrow b^2 - 4ac \geq 0$

different $\Rightarrow b^2 - 4ac \neq 0$

positive $\Rightarrow b < 0$ & $b < \sqrt{b^2 - 4ac}$

3.2.

#3

$$W = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{vmatrix} = e^{-2t}(e^{-2t} - 2te^{-2t}) - te^{-2t}(-2e^{-2t})$$

$$= e^{-4t} - \cancel{2te^{-4t}} + \cancel{2te^{-4t}} = e^{-4t} \neq 0$$

6.

$$W = \begin{vmatrix} \cos^2\theta & 1 + \cos 2\theta \\ -2\cos\theta\sin\theta & -2\sin 2\theta \end{vmatrix} = \begin{vmatrix} \frac{1}{2}(1 + \cos 2\theta) & 1 + \cos 2\theta \\ -\sin 2\theta & -2\sin 2\theta \end{vmatrix} =$$

8 cont'd

$$-(1+\cos 2\theta)\sin 2\theta + (1+\cos 2\theta)(+\sin 2\theta) = 0$$

not a fundamental set

9. $$\frac{\cancel{t(t-4)}\,y''}{\cancel{t(t-4)}} - \frac{3\cancel{t}y'}{\cancel{t}(t-4)} + \frac{4y}{t(t-4)} = \frac{\sin t}{t(t-4)}$$

$$y'' - \frac{3}{t-4}y' + \frac{4}{t(t-4)}y = \frac{\sin t}{t(t-4)}$$

potential problems at $t=4,\ t=0$

by Abel's Theorem

$$W = e^{+\int +\frac{3}{t-4}\,dt} = e^{3\ln|t-4|} = e^{\ln(t-4)^3} = (t-4)^3 \neq 0$$

$y(3)=0,\ y'(3)=-1 \Rightarrow (0,4)$

16. $y = \sin t^2$ $\quad y' = 2t\cos t^2$ $\quad y'' = -2t\sin t^2 \cdot 2t + 2\cos t^2$

$-4t^2\sin t^2 + 2\cos t^2$

$$y'' + p(t)y' + q(t)y = 0$$

$$-4t^2\sin t^2 + 2\cos t^2 + p(t)\,2t\cos t^2 + q(t)\sin t^2 = 0$$

$(\sin t^2)(-4t^2 + q(t)) = 0 \qquad \cos t^2(2 + p(t)\cdot 2t) = 0$

$q(t) = 4t^2 \qquad 2 + 2tp(t) = 0$

$p(t) = -\frac{2}{2t} = -\frac{1}{t}$

no, since $p(t)$ would have to be $-\frac{1}{t}$ which is not continuous at $t=0$

22. $y'' + y' - 2y = 0 \qquad r^2 + r - 2 = 0$

$(r+2)(r-1) = 0 \qquad r=-2,\ r=1$

$y_1 = e^{-2t} \qquad y_2 = e^t$

$$W = \begin{vmatrix} e^{-2t} & e^t \\ -2e^{-2t} & e^t \end{vmatrix} = e^{-t} + 2e^{-t} = 3e^{-t} \neq 0$$

yes, it is a fundamental set

25. $y'' - 2y' + y = 0$

$y_1(t) = e^t \qquad y_2(t) = te^t$

$y_1' = e^t \qquad y_2' = e^t + te^t$

$y_1'' = e^t \qquad y_2'' = e^t + e^t + te^t = 2e^t + te^t$

y_1: $e^t - 2e^t + e^t = 0$ ✓

y_2: $2e^t + te^t - 2(e^t + te^t) + te^t =$

$\cancel{2e^t} + \cancel{te^t} - \cancel{2e^t} - \cancel{2te^t} + \cancel{te^t} = 0$ ✓

$$W = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t} + \cancel{te^{2t}} - \cancel{te^{2t}} = e^{2t} \neq 0$$

yes, they do constitute a fundamental set

29. $t^2 y'' - t(t+2)y' + (t+2)y = 0$

$$y'' - \frac{\cancel{t}(t+2)}{t^{\cancel{2}}} y' + \frac{t+2}{t^2} y = 0$$

$$p(t) = -\frac{t+2}{t} = -1 - \frac{2}{t}$$

$$W = e^{-\int -1 - \frac{2}{t}\,dt} = e^{t + 2\ln t} = e^t \cdot e^{\ln t^2} = t^2 e^t$$

$= 0$ when $t = 0$ only