Homework #1 Key (Math 285 Spring 2012)

1.1

4. $y' = -1 - 2y$ See attached graphs

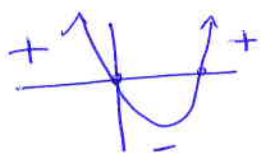
stationary point $0 = -1 - 2y \Rightarrow 1 = -2y \quad y = -\frac{1}{2}$

this point is attracting as $t \to \infty$

12. $y' = -y(5-y) = y(y-5)$ See attached graphs

Stationary point $0 = y(y-5)$

$y = 0, \ y = 5$



$y=0$ is attracting $y = 5$ is repelling

for $y_0 < 5$ approach $y = 0$ as $t \to \infty$

for $y_0 > 5$ $y \to \infty$ as $t \to \infty$

15. j
16. c.
17. g
18. b.
19. h
20. e

31. $y' = 2t - 1 - y^2$ See attached graphs

Stationary point $0 = 2t - 1 - y^2 \Rightarrow y^2 = 2t - 1$

$y = \pm\sqrt{2t-1}$

the top $\sqrt{\ }$ is attracting, the $-\sqrt{\ }$ is repelling

1.2

26. $\frac{dy}{dt} = 2y-5 \qquad y(0)=y_0$

$\frac{dy}{dt} = 2(y-5/2) \Rightarrow \int \frac{dy}{y-5/2} = \int 2\,dt \Rightarrow \ln|y-5/2| = 2t+C$

$y - 5/2 = Ae^{2t} \Rightarrow y = Ae^{2t} + 5/2$

$y_0 = Ae^{2(0)} + 5/2 \Rightarrow (y_0 - 5/2) = A$

$y = (y_0 - 5/2)e^{2t} + 5/2$ ($5/2 = y$ is the stationary point)

on graphing page, solutions plotted for $y_0 =$
$5,\ 3,\ 5/2,\ 1,\ 0$

5. a) $\frac{dy}{dt} = ay \Rightarrow \int \frac{dy}{y} = \int a\,dt \Rightarrow \ln y = at + C \Rightarrow$

$y_1 = Ae^{at}$

b) $y = Ae^{at} + k$

$y' = Aae^{at}$

$\frac{dy}{dt} = ay - b \Rightarrow Aae^{at} = a(Ae^{at} + k) - b$

$\cancel{Aae^{at}} = \cancel{Aae^{at}} + ak - b$

$0 = ak - b$

$k = b/a \Rightarrow \boxed{y = Ae^{at} + b/a}$

c) $\frac{dy}{dt} = ay - b \Rightarrow \frac{dy}{dt} = a(y - b/a) \Rightarrow \int \frac{dy}{y - b/a} = \int a\,dt$

$\ln|y - b/a| = at + C \Rightarrow Ae^{at} = y - b/a \Rightarrow \boxed{y = Ae^{at} + b/a}$

1.2.

9. $\frac{dv}{dt} = 9.8 - \frac{v}{5} \Rightarrow \frac{dv}{dt} = \frac{1}{5}(49 - v) \Rightarrow \int \frac{dv}{49-v} = \int \frac{1}{5} dt \Rightarrow$

$v(0)=0$

$-\ln|49-v| = \frac{1}{5}t + C \Rightarrow \ln|49-v| = -\frac{1}{5}t + C \Rightarrow$

$49 - v = Ae^{-\frac{1}{5}t} \Rightarrow v = 49 - Ae^{-\frac{1}{5}t}$

$0 = 49 - Ae^{0} \Rightarrow A = 49$

$v = 49 - 49e^{-\frac{1}{5}t}$

49 is the limiting velocity

98% of 49 = 48.02 m/sec

a) $48.02 = 49 - 49e^{-\frac{1}{5}t}$

$.02 = e^{-\frac{1}{5}t} \Rightarrow \ln .02 = -\frac{1}{5}t \Rightarrow t = -5\ln .02 = 19.56$ sec.

b) $\int v\,dt$ = position

$\int_0^{19.56} 49 - 49e^{-\frac{1}{5}t}\,dt = 49t + 245e^{-\frac{1}{5}t}\Big|_0^{19.56} = 718.34$ m.

14. half-life = 1620 years.

reduced by 1/4 ⇒ 3/4 are left

$\frac{1}{2}A = Ae^{k(1620)} \Rightarrow \frac{1}{2} = e^{1620k} \Rightarrow \frac{\ln \frac{1}{2}}{1620} = \cancel{1620}k$

$k = -4.2786863 \times 10^{-4}$

$\frac{3}{4}A = Ae^{-4.278\ldots \times 10^{-4}t} \Rightarrow \frac{3}{4} = e^{-4.278 \times 10^{-4}t} \Rightarrow$

$\ln(3/4) = -4.278\ldots \times 10^{-4}t \Rightarrow t = 672.4$ years

1.2.

16. $\frac{du}{dt} = -k(u-T)$ $\quad k = .15h^{-1}$

$T = 10°F$

$u(0) = 70°F$

$\int \frac{du}{u-T} = \int -k\,dt$

$\ln|u-T| = -kt + C \Rightarrow u-T = Ae^{-kt} \Rightarrow u = Ae^{-kt} + T$

$u = Ae^{-.15t} + 10° \Rightarrow 70 = Ae^{0} + 10° \Rightarrow A = 60$

$u = 60e^{-.15t} + 10°$

$32° = 60e^{-.15t} + 10° \Rightarrow \frac{22}{60} = \frac{60}{60}e^{-.15t} \Rightarrow 11/30 = e^{-.15t} \Rightarrow$

$\ln(11/30) = -.15t \Rightarrow \frac{\ln(11/30)}{-.15} = t \Rightarrow t = 6.68868\ldots$

≈ 6.7 hours.

18. a) $\frac{dQ}{dt} = \frac{.01\,g}{gal} \cdot \frac{300\,gal}{hr} - \frac{300\,gal}{hr} \cdot \frac{Q}{1{,}000{,}000\,gal}$

$\frac{dQ}{dt} = 3 - 3\times10^{-4}Q \Rightarrow \frac{dQ}{dt} = 3\times10^{-4}(10^4 - Q) \Rightarrow$

$\int \frac{dQ}{10^4 - Q} = \int 3\times10^{-4}\,dt \Rightarrow -\ln|10^4 - Q| = 3\times10^{-4}t + C$

$\ln|10^4 - Q| = -3\times10^{-4}t + C \Rightarrow 10^4 - Q = Ae^{-3\times10^{-4}t} \Rightarrow$

$Q = 10^4 - Ae^{-3\times10^{-4}t}$

$Q(0) = 0 \Rightarrow 0 = 10^4 - Ae^{0} \Rightarrow A = 10^4$

$Q = 10^4 - 10^4e^{-3\times10^{-4}t}$

5. cont'd

$e^{-2t}y' - 2e^{-2t}y = 3e^{-t}$

$\int (e^{-2t}y)' = \int 3e^{-t} \Rightarrow e^{-2t}y = -3e^{-t} + C$

$y = -3e^{t} + Ce^{2t}$

10. $\dfrac{ty' - y = t^2 e^{-t}}{t} \Rightarrow y' - \frac{1}{t}y = te^{-t}$

$\mu(t) = e^{\int -\frac{1}{t}dt} = e^{-\ln t} = \frac{1}{t}$

$\frac{1}{t}y' - \frac{1}{t^2}y = e^{-t}$

$\int \left(\frac{1}{t}y\right)' = \int e^{-t} \Rightarrow \frac{1}{t}y = -e^{-t} + C$

$y = -te^{-t} + Ct$

See attached graphs

15. $ty' + 2y = t^2 - t + 1 \qquad y(1) = \frac{1}{2}$

$y' + \frac{2}{t}y = t - 1 + \frac{1}{t}$

$\mu(t) = e^{\int \frac{2}{t}dt} = e^{2\ln t} = t^2$

$t^2y' + 2ty = t^3 - t^2 + t \Rightarrow \int (t^2y)' = \int t^3 - t^2 + t \, dt$

$t^2y = \frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2 + C$

$y = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + Ct^{-2}$

$\cancel{\frac{1}{2}} = \frac{1}{4} - \frac{1}{3} + \cancel{\frac{1}{2}} + C \Rightarrow C = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

$y = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{1}{12t^2}$

34. Solutions have the limit 3 as $t \to \infty$

$y' = y - 3 \implies y' - y = -3$

36. Solutions have the limit $y = 2t - 5$ as $t \to \infty$

$0 = 2t - 5 - y$

$y' = 2t - 5 - y$

$y' + y = 2t - 5$

2.2.

2. $y' = \dfrac{x^2}{y(1+x^3)} \implies \int y\,dy = \int \dfrac{x^2}{(1+x^3)}\,dx$

$\dfrac{y^2}{2} = \frac{1}{3}\ln|1+x^3| + C \implies$

$y^2 = \ln(1+x^3)^{2/3} + C$

$y = \pm\sqrt{\ln(1+x^3)^{2/3} + C}$

4. $y' = \dfrac{3x^2-1}{3+2y} \implies \int \dfrac{dy}{3+2y} = \int 3x^2 - 1\,dx$

$\frac{1}{2}\ln|3+2y| = x^3 - x + C$

$\ln|3+2y| = 2x^3 - 2x + C$

$3 + 2y = Ae^{2x^3-2x}$

$y = Ae^{2x^3-2x} - 3/2$

13. $y' = \dfrac{2x}{y(1+x^2)} \implies \int y\,dy = \int \dfrac{2x\,dx}{1+x^2}$

$\dfrac{y^2}{2} = \ln|1+x^2| + C$

$y^2 = \ln(1+x^2)^2 + C$

$y = \pm\sqrt{\ln(1+x^2)^2 + C}$

$y(0) = -2$

$-2 = \pm\sqrt{\ln(1) + C}$

$y = -\sqrt{\ln(1+x^2)^2 + 4}$

19. $\sin 2x\,dx + \cos 3y\,dy = 0$ $\qquad y(\pi/2) = \pi/3$

$\int \cos 3y\,dy = -\int \sin 2x\,dx$

$\frac{1}{3}\sin 3y = \frac{1}{2}\cos 2x + C$

$\sin 3y = \frac{3}{2}\cos 2x + C$ $\qquad \sin \pi = \frac{3}{2}\cos \pi + C$

$0 = \frac{3}{2}(-1) + C$

$y = \frac{1}{3}\arcsin\left[\frac{3}{2}\cos 2x + C\right]$ $\qquad C = +3/2$

$y = \frac{1}{3}\arcsin\left[3/2\cos 2x + 3/2\right]$

21. $y' = \dfrac{(1+3x^2)}{3y^2-6y}$ $\qquad y(0) = 1$

$\int 3y^2 - 6y\,dy = \int 1 + 3x^2\,dx \Rightarrow \quad y^3 - 3y^2 = x + x^3 + C$

$1 - 3 = C \quad \Rightarrow C = -2$

$y^3 - 3y^2 = x^3 + x - 2$

both sides are defined for all real numbers.

but y' is not defined when $3y^2 - 6y = 0$

$3y(y-2) = 0$

$y = 0$ and when $y = 2$

31. $\dfrac{dy}{dx} = \dfrac{x^2 + xy + y^2}{x^2}$ $\qquad y = vx$ $\qquad \dfrac{dy}{dx} = \dfrac{dv}{dx}\cdot x + v$

all terms degree 2 $\Rightarrow$ homogeneous degree 0 (cancels)

$\dfrac{dv}{dx}\cdot x + v = \dfrac{x^2 + xvx + v^2x^2}{x^2}$

See attached graphs

$\dfrac{dv}{dx}\cdot x + \cancel{v} = 1 + \cancel{v} + v^2 \quad \Rightarrow \int \dfrac{dv}{1+v^2} = \int \frac{1}{x}\,dx$

$\arctan v = \ln x + C = \ln Ax$

$v = \tan(\ln(Ax)) \quad \Rightarrow \quad y = x\tan(\ln Ax)$

36. $(x^2+3xy+y^2)\,dx - x^2\,dy = 0$

$(x^2+3xy+y^2)\,dx = x^2\,dy$

all terms degree 2
$\Rightarrow$ homogeneous degree 0

$$\frac{x^2+3xy+y^2}{x^2}\,dx = dy \qquad y = vx, \quad dy = dv\cdot x + dx\cdot v$$

$$\frac{x^2+3x^2v+v^2x^2}{x^2}\,dx = dv\cdot x + v\,dx$$

$$(1+3v+v^2-v)\,dx = x\,dv$$

$$\int \frac{1}{x}\,dx = \frac{dv}{1+2v+v^2} = \int \frac{dv}{(1+v)^2}$$

$$\ln Ax = \ln x + C = -\frac{1}{1+v}$$

$$1+v = \frac{-1}{\ln Ax} \Rightarrow v = -\frac{1}{\ln Ax} - 1$$

$$y = \frac{-x}{\ln Ax} - x$$

See attached graphs

37. $y' = \dfrac{x^2-3y^2}{2xy}$

all terms degree 2 $\Rightarrow$ homogeneous degree 0

$$y = vx \Rightarrow y' = v'x + v$$

$$v'x + v = \frac{x^2-3v^2x^2}{2x^2v} \Rightarrow v'x + v = \frac{1}{2}\left(\frac{1-3v^2}{v}\right) - v$$
$$-v$$

$$v'x = \frac{1}{2}\left(\frac{1-3v^2-2v^2}{v}\right) = \frac{1}{2}\left(\frac{1-5v^2}{v}\right)$$

$$\int \frac{v\,dv}{1-5v^2} = \int \frac{1}{2x}\,dx \Rightarrow -\frac{1}{10}\ln|1-5v^2| = \frac{1}{2}\ln x + C$$

$$\ln|1-5v^2| = -5\ln x + C$$

$$1-5v^2 = \frac{A}{x^5}$$

$$1-\frac{A}{x^5} = 5v^2$$

$$\pm\sqrt{\frac{1}{5}-\frac{A}{5x^5}} = v$$

$$y = \pm x\sqrt{\frac{1}{5}-\frac{A}{5x^2}}$$

See graphing attached

4. $\frac{dQ}{dt} = \text{rate in} - \text{rate out}$

$\text{rate out} = \frac{Q}{200+t}(2)$ lbs/min $\qquad Q(0) = 100$

$\text{rate in} = 3 \cdot 1$ lbs/min

$$\frac{dQ}{dt} = 3 - \frac{2Q}{200+t} \Rightarrow Q' + \frac{2}{200+t} Q = 3$$

$$\mu(t) = e^{\int \frac{2}{200+t} dt} = e^{2\ln(200+t)} = e^{\ln(200+t)^2} = (200+t)^2$$

$$(200+t)^2 Q' + 2(200+t) Q = 3(200+t)^2$$

$$\int \left((200+t)^2 Q \right)' = \int 3(200+t)^2 \, dt$$

$$(200+t) Q = (200+t)^3 + C$$

$$Q = (200+t)^2 + \frac{C}{200+t}$$

$$100 = (200)^2 + \frac{C}{200}$$

$$C = -7.98 \times 10^6$$

8. $S(0) = 0$

a) $\frac{dS}{dt} = rS + k \Rightarrow \frac{dS}{dt} = r\left(S + \frac{k}{r}\right)$

$$\int \frac{dS}{S + \frac{k}{r}} = \int r \, dt$$

$$\ln\left|S + \frac{k}{r}\right| = rt + C$$

$$S + \frac{k}{r} = Ae^{rt} \Rightarrow S = Ae^{rt} - \frac{k}{r}$$

8. Cont'd

b) $1{,}000{,}000 = Ae^{.075(40)} - \frac{k}{.075}$

$0 = Ae^0 - \frac{k}{.075} \Rightarrow A = \frac{k}{.075}$

$1.0\times10^6 = \frac{k}{.075} e^{.075(40)} - \frac{k}{.075}$

$75{,}000 = k(e^{.075(40)} - 1) \Rightarrow k = \frac{75{,}000}{e^{.075(40)} - 1}$

$k = 3929.68$

$A = 52{,}395.7$

c) $S = Ae^{rt} - \frac{k}{r} \Rightarrow S = Ae^{rt} - \frac{2000}{r}$

$S(0) = 0 \quad 0 = Ae^{r(0)} - \frac{2000}{r} \Rightarrow A = \frac{2000}{r}$

$S(40) = 10^6 \quad 10^6 = \frac{2000}{r} e^{40r} - \frac{2000}{r}$

$500 = \frac{1}{r}(e^{40r} - 1)$

Solve numerically

$r = .097734 \sim 9.77\%$

$A = 20{,}463.7$