Name KEY

Math 285, Exam #3, Spring 2012

Instructions: Show all work. You may use a TI-89 to check your answers, but points will be deducted if you obtain answers solely from your calculator. Be sure to answer all parts of all the questions, and use exact answers unless directed to round, or in word problems.

1. Find the radius and interval of convergence of the infinite series $\sum_{n=1}^{\infty} \frac{(n+3)(x-1)^{n+1}}{n^2 3^n}$ (10 points)

$$\lim_{n\to\infty} \left| \frac{(n+4)(x-1)^{n+2}}{(n+1)^2 3^{n+1}} \cdot \frac{n^2 3^n}{(n+3)(x-1)^{n+1}} \right| = \lim_{n\to\infty} \left| \cancel{\frac{(n+4)n^2}{(n+1)^2(n+3)}} \right|^{\to 1} \lim_{n\to\infty} \left| \frac{x-1}{3} \right| < 1$$

$$\left| \frac{x-1}{3} \right| < 1 \Rightarrow -1 < \frac{x-1}{3} < 1 \Rightarrow -3 \leq x-1 \leq 3$$

$$+1$$

$$-2 \leq x \leq 4$$

$\boxed{\text{radius of convergence} = 3 \qquad \text{interval of convergence } [-2, 4)}$

endpoints @ $x=-2$ $\sum \frac{(n+3)(-3)^{n+1}}{n^2 3^n} = \sum \frac{(n+3)(-1)^{n+1}(3)}{n^2}$ converges / alternating

@ $x=4$ $\sum \frac{(n+3)3^{n+1}}{n^2 3^n} = \sum \frac{(n+3)3}{n^2}$ diverges by comparison w/ p-series

2. Write the function $f(x) = \sin(2x)$ as a Taylor series centered at x=0. (7 points)

$$\sin x = \sum_{n=0} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$\sin 2x = \sum_{n=0} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} = \sum \frac{(-1)^n 2^{2n+1} x^{2n+1}}{(2n+1)!}$$

3. Given the power series $y = \sum_{n=0}^{\infty} \frac{a_n}{n+2}(x-2)^n$, write the first 2 derivatives as power series that starts at n=0. (10 points)

$$y' = \sum_{n=1}^{\infty} \frac{a_n}{n+2} n(x-2)^{n-1} = \sum_{n=0}^{\infty} \frac{a_{n+1}}{n+3}(n+1)(x-2)^n$$

$$y'' = \sum_{n=2}^{\infty} \frac{a_n}{n+2} n(n-1)(x-2)^{n-2} = \sum_{n=0}^{\infty} \frac{a_{n+2}}{n+4}(n+2)(n+1)(x-2)^n$$

4. Determine if the following functions are even, odd or neither. Based on that information, will the Fourier series representing the function have only cosine terms, only sine terms, or possibly both? (3 points each)

a. $p(x) = e^{-x^2}$

even, cosine only

b. s(x)=sinh(9x)

odd, sine only

c. $q(x) = e^x$

neither both

d. $h(x) = \dfrac{x}{x^2-1}$ $\quad \frac{\text{odd}}{\text{even}} = \text{odd}$

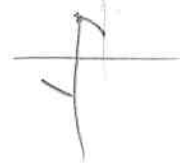
odd, sine only

5. Extend the function $f(x) = 4 - x^2, 0 < x < 1$ as the indicated type of Fourier series. State the piecewise definition of the periodic piece on the appropriate interval (-L,L) and state the resulting period. Sketch a graph of the resulting extension. (6 points each)

a. As a sine (only) series

$\Rightarrow$ odd

$4 - x^2$



$$f(x) = \begin{cases} x^2 - 4 & -1 \le x < 0 \\ 4 - x^2 & 0 \le x < 1 \end{cases}$$

$L = 1$

period $= 2L = 2$

b. As a cosine (only) series

$f(x) = 4 - x^2, \quad -1 < x < 1$

(graph is naturally even)

$L = 1$

period $= 2L = 2$

c. Set the interval (-L,0) to be zero. What kind of Fourier series would result?

$L = 1$

$$f(x) = \begin{cases} 0 & -1 \le x < 0 \\ 4 - x^2 & 0 \le x \le 1 \end{cases}$$

period $= 2L = 2$

Fourier Series would contain both sine & cosine terms as this function is neither even or odd

6. Use power series to solve the equation $y'' + 4y = 0, x_0 = 0$ centered at the designated point. Solve the coefficients in terms of a_0 and a_1. Is the solution a fundamental set? Justify your answer by using the Wronskian. (25 points)

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} a_n x^{n-1} \cdot n = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + 4 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} x^n \left((n+2)(n+1) a_{n+2} + 4 a_n \right) = 0$$

$a_0 / n=0$ $\qquad$ $a_1 / n=1$

$$\boxed{\cancel{(n+2)(n+1)}\, a_{n+2} = \frac{-4 a_n}{(n+2)(n+1)}}$$

$n=0$

$$a_2 = \frac{-4a_0}{2 \cdot 1} = \frac{(-1) 2^2 a_0}{2!}$$

$$a_4 = \frac{-4a_2}{3 \cdot 4} = \frac{16 a_0}{4!} = \frac{(-1)^2 2^4 a_0}{4!}$$

$$a_6 = \frac{-4a_4}{5 \cdot 6} = \frac{(-1)^3 2^6 a_0}{6!}$$

$\vdots$

$n=1$

$$a_3 = \frac{-4a_1}{3 \cdot 2} = \frac{(-1) 2^2 a_1}{3!}$$

$$a_5 = \frac{-4a_3}{4 \cdot 5} = \frac{(-1)^2 2^4 a_1}{5!}$$

$$a_7 = \frac{-4a_5}{6 \cdot 7} = \frac{(-1)^3 2^6 a_1}{7!}$$

$$y(t) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n+1}}{(2n+1)!}$$

$\cos(2x)$ $\qquad$ $\sin(2x)$

$$W = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$ yes, a fundamental set

7. Solve the eigenvalue problem $y'' - \lambda y = 0, y(0) = 0, y'(L) = 1$ for the values of λ that give a nontrivial solution to the equation. Check $\lambda>0$, $\lambda=0$, and $\lambda<0$. Find the corresponding eigenfunctions for each value of λ. (25 points)

$\lambda > 0 \quad \lambda = \mu^2$

$y'' - \mu^2 y = 0$

$r^2 - \mu^2 = 0$

$r^2 = \mu^2$

$r = \pm\mu$

$y = Ae^{-\mu t} + Be^{\mu t}$

$0 = A + B \qquad A = -B$

$y' = -\mu A e^{-\mu t} + \mu B e^{\mu t}$

$1 = -\mu A e^{-\mu L} + \mu B e^{\mu L}$

$1 = +\mu B e^{-\mu L} + \mu B e^{\mu L}$

$\left(0 = \mu B e^{-\mu L} + \mu B e^{\mu L} - 1\right) e^{\mu L}$

$0 = \mu B + \mu B e^{2\mu L} - e^{\mu L}$

let $u = e^{\mu L}$

$\mu B u^2 - u + \mu B = 0$

$$u = \frac{1 \pm \sqrt{1^2 - 4(\mu B)(\mu B)}}{2\mu B} = e^{\mu L}$$

$$\frac{1 \pm \sqrt{1 - 4\mu^2 B^2}}{2\mu B} = e^{\mu L}$$

if $1 - 4\mu^2 B^2 \geq 0$

$$\boxed{B = \frac{1}{\mu e^{-\mu L} + \mu e^{\mu L}}}$$

unique for each μ (rewrite as cosh/sinh)

$1 - 4\mu^2 B^2 \geq 0$

$1 > 4\mu^2 B^2$

$\lambda < 0 \quad \lambda = -\mu^2$

$y'' + \mu^2 y = 0$

$r^2 = \pm \mu i$

$y = \cancel{A\cos\mu t} + B\sin\mu t$

$0 = A$

$y' = B\cos\mu t$

$1 = B\cos\mu L$

$\frac{1}{B} = \cos\mu L$

$\cos^{-1}\left(\frac{1}{B}\right) = \mu L$

$\boxed{\frac{1}{L}\cos^{-1}\left(\frac{1}{B}\right) = \mu}$ unique for each μ

$1/B = \mu e^{-\mu L} + \mu e^{\mu L}$

$\lambda = 0$

$y'' = 0$

$y = Ax + B$

$0 = A$

$y' = 0$

no solution cannot satisfy boundary conditions for $\lambda = 0$

$\lambda > 0$

$y = Ae^{-\mu t} + Be^{-\mu t}$

$A = -B, \; B = \dfrac{1}{\mu e^{-\mu L} + \mu e^{\mu L}}$

$\lambda < 0$

$y = B\sin\mu t$

$B \doteq \sec\mu L$

unique for each μ

$\lambda \neq 0$

8. For problem #7: (5 points each)
   a. Describe the solution: Is it finite or infinite in number? Is it unique or have no solution, trivial-only solution, etc.

for $\lambda > 0$ ∃ $x < 0$ each value of λ has a unique solution (finite)

$\lambda = 0$ has no solution

   b. Is the problem homogeneous or non-homogeneous?

non-homogeneous

9. Solve the boundary value problem $y'' + 2y = x, y(0) = 0, y(\pi) = 0$. (12 points)

$r^2 + 2 = 0$

$r = \pm\sqrt{2}\,i$

$Y_c = C\cos\sqrt{2}x + D\sin\sqrt{2}x$

$Y(x) = Ax + B$

$Y' = A$

$Y'' = 0$

$0 + 2(Ax + B) = x$

$B = 0$

$2A = 1$

$A = \frac{1}{2}$

$y(x) = \underline{C\cancel{\cos\sqrt{2}x}} + D\sin\sqrt{2}x + \frac{1}{2}x$

$0 = C \quad + \quad 0 \quad + \quad 0$

$C = 0$

$0 = D\sin\sqrt{2}\pi + \frac{1}{2}\pi$

$D\sin\sqrt{2}\pi = -\frac{1}{2}\pi$

$D = -\frac{1}{2}\pi / \sin\sqrt{2}\pi$

note: $\sin\sqrt{2}\pi = \#$

$\approx -.9639$

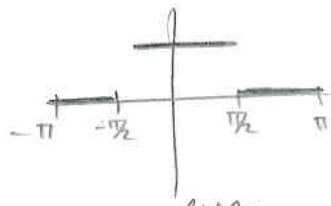
unique solution

$\underline{y(x) = -\frac{1}{2}\pi / \sin(\sqrt{2}\pi)\sin\sqrt{2}x + \frac{1}{2}x}$

10. Find the Fourier series for the periodic function

$$f(x)=\begin{cases}0 & -\pi\le x<-\frac{\pi}{2}\\ 1 & -\frac{\pi}{2}\le x<\frac{\pi}{2}\\ 0 & \frac{\pi}{2}\le x<\pi\end{cases}\qquad f(x+2\pi)=f(x)\ \text{(25 points)}$$

$$f(x)=\frac{a_0}{2}+\sum_{m=1}^{\infty}\left(a_m\cos\frac{m\pi x}{L}+b_m\sin\frac{m\pi x}{L}\right)$$



even

$$a_m=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos m\frac{\pi x}{L}\,dx=$$

$$a_0=\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}1\,dx=\frac{1}{\pi}\cdot\pi=1$$

$$\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}1\cos\frac{m\cancel{\pi}x}{\cancel{\pi}}\,dx$$

$$\frac{1}{\pi}\cdot\frac{\ }{m}\sin\frac{m\ x}{\pi}\Big|_{-\pi/2}^{\pi/2}$$

$$f(x)=\frac{1}{2}+\sum\frac{2}{(2k+1)\pi}(-1)^k\cos\left(\frac{(2k+1)\pi x}{\pi}\right)$$

$$\frac{1}{m\pi}\left[\sin\frac{m\pi}{2}+\sin\frac{m\pi}{2}\right]=$$

$$\boxed{f(x)=\frac{1}{2}+\sum_{k=0}^{\infty}\frac{2(-1)^k}{(2k+1)\pi}\cos[(2k+1)x]}$$

$$\frac{1}{m\pi}2\sin\frac{m\pi}{2}=$$

m even
$m=2k$
$\sin k\pi=0$

m odd
$m=(2k+1)$
$\sin\left(\frac{2k\pi}{2}+\frac{\pi}{2}\right)$
reduces to alternating
1, -1, 1, -1...

only odd cosines remain

$$a_m=\frac{2}{m\pi}$$

$$b_m=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin\frac{m\cancel{\pi}x}{\cancel{\pi}}\,dx$$

f(x) is even so this = 0

11. For the differential equation $y'' + xy' + 2y = 0, x_0 = 0$, use a power series to solve the system. Simplify the expression until you can factor out a factor x^n and all the series start at the same value of the index, then stop. (7 points)

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty} a_{n+1}(n+1)x^n$$

$$y'' = \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n$$

from #6

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + x\sum_{n=0}^{\infty} a_{n+1}(n+1)x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=0}^{\infty} a_{n+1}(n+1)x^{n+1} + \sum_{n=0}^{\infty} 2a_n x^n = 0$$

$$\sum_{n=1}^{\infty} a_n n x^n$$

$$\sum_{n=1}^{\infty} a_{n+2}(n+2)(n+1)x^n + a_2(2)(1)(1) + \sum_{n=1}^{\infty} a_n n x^n + \sum_{n=1}^{\infty} 2a_n x^n + 2a_0 = 0$$

$$2a_2 + 2a_0 + \sum_{n=1}^{\infty}\left[a_{n+2}(n+2)(n+1) + a_n(2+n)\right]x^n = 0$$

$a_0 = 0$

$a_2 = 0$

$$a_{n+2}(n+2)(n+1) + a_n(2+n) = 0$$