* marks a possible coefficient error

Name KEY

Math 285, Exam #2, Spring 2012

Instructions: Show all work. You may use a TI-89 to check your answers, but points will be deducted if you obtain answers solely from your calculator. Be sure to answer all parts of all the questions, and use exact answers unless directed to round, or in word problems.

1. Solve the homogeneous second order differential equations with constant coefficients. Other problems will refer back to these problems, so don't rush! Be sure to clearly indicate the characteristic equation in each case. Solve for any variables. (10 points each)

* a. $y'' + 8y' - 9y = 0, y(1) = 1, y'(1) = 0$

$r^2 + 8r - 9 = 0$ $\quad y_1 = e^{-9t} \quad y_2 = e^t$

$(r+9)(r-1) = 0$

$r = -9 \quad r = 1$

$y = Ae^{-9t} + Be^t \qquad y' = -9Ae^{-9t} + Be^t$

$1 = Ae^{-9} + Be \qquad 0 = -9Ae^{-9} + Be \Rightarrow B = \frac{9Ae^{-9}}{e} \Rightarrow B = 9Ae^{-10}$

$\approx 76.803 \Rightarrow B = \frac{9}{e+9}$

$1 = Ae^{-9} + 9Ae^{-10}$

$1 = A(e^{-9} + 9e^{-10}) \Rightarrow A = \frac{1}{e^{-9} + 9e^{-10}} = \frac{1}{e^{-9}(1 + 9e^{-1})} = \frac{e^9}{1 + 9/e} = \frac{e^{10}}{e+9} \approx 1879.67$

$$\boxed{y(t) = \left(\frac{e^{10}}{e+9}\right)e^{-9t} + \left(\frac{9}{e+9}\right)e^t}$$

b. $y'' - 2y' + 4y = 0, y(0) = 1, y'(0) = 0$

$r^2 - 2r + 4 = 0 \qquad r = \frac{2 \pm \sqrt{4 - 16}}{2} = \frac{2 \pm \sqrt{-12}}{2} = \frac{2 \pm 2\sqrt{3}i}{2}$

$r = 1 \pm \sqrt{3}i$

$y_1 = e^t \cos\sqrt{3}t \qquad y_2 = e^t \sin\sqrt{3}t$

$y = Ae^t \cos\sqrt{3}t + Be^t \sin\sqrt{3}t \qquad y' = Ae^t \cos\sqrt{3}t - A\sqrt{3}e^t \sin\sqrt{3}t + Be^t \sin\sqrt{3}t + B\sqrt{3}e^t \cos\sqrt{3}t$

$1 = A + 0 \qquad A = 1$

$0 = A + \sqrt{3}B \qquad 1 = -\sqrt{3}B$

$B = -1/\sqrt{3}$

$$\boxed{y(t) = e^t \cos\sqrt{3}t - \tfrac{1}{\sqrt{3}}e^t \sin\sqrt{3}t}$$

c. $9y'' + 24y' + 16y = 0, y(0) = 2, y'(0) = -1$

$9r^2 + 24r + 16 = 0$

$(3r+4)^2 = 0 \qquad r = -4/3 \qquad y_1 = e^{-4/3t} \qquad y_2 = te^{-4/3t}$

$y = Ae^{-4/3t} + Bte^{-4/3t} \qquad y' = -4/3Ae^{-4/3t} + Be^{-4/3t} + -4/3Bte^{-4/3t}$

$2 = A \qquad -1 = -4/3A + B$

$-1 = -8/3 + B$

$5/3 = B$

$\boxed{y(t) = 2e^{-4/3t} + 5/3te^{-4/3t}}$

2. Find the value of the Wronskian for $y'' + (\cos t)y' + 3(\ln|t|)y = 0, y(2) = 3, y'(2) = 1$ using Abel's Theorem without solving the system. Determine the largest interval on which the solution exists, given the initial conditions. (8 points)

$W = e^{-\int p(t)\,dt} = e^{-\int \cos t\,dt} = \boxed{e^{-\sin t}}$

W defined everywhere

Cost defined everywhere

ln|t| defined everywhere but 0

initial conditions at 2 so $(0, \infty)$

3. For each of the solutions in Problem #1 (a-c), find the value of the Wronskian and show that the set of solutions you found forms a fundamental set. (12 points)

a. $y_1 = e^{-9t}$ $y_2 = e^{t}$

$$W = \begin{vmatrix} e^{-9t} & e^{t} \\ -9e^{-9t} & e^{t} \end{vmatrix} = e^{-8t} + 9e^{-8t} = 10e^{-8t}$$

defined for all t
fundamental set

b. $y_1 = e^{t}\cos\sqrt{3}t$ $y_2 = e^{t}\sin\sqrt{3}t$

$$W = \begin{vmatrix} e^{t}\cos\sqrt{3}t & e^{t}\sin\sqrt{3}t \\ e^{t}\cos\sqrt{3}t - \sqrt{3}e^{t}\sin\sqrt{3}t & e^{t}\sin\sqrt{3}t + \sqrt{3}e^{t}\cos\sqrt{3}t \end{vmatrix} =$$

$$e^{2t}\cancel{\sin\sqrt{3}t}\,\cos\sqrt{3}t + \sqrt{3}e^{2t}\cos^2\sqrt{3}t - e^{2t}\cancel{\sin\sqrt{3}t\cos\sqrt{3}t} + \sqrt{3}e^{2t}\sin^2\sqrt{3}t =$$

$$\sqrt{3}e^{2t}(\cos^2\sqrt{3}t + \sin^2\sqrt{3}t) = \sqrt{3}e^{2t}$$ ✓

defined for all t, fundamental set

c. $y_1 = e^{-4/3t}$ $y_2 = te^{-4/3t}$

$$W = \begin{vmatrix} e^{-4/3t} & te^{-4/3t} \\ -4/3e^{-4/3t} & e^{-4/3t} - 4/3te^{-4/3t} \end{vmatrix} = e^{-8/3t} - \cancel{4/3te^{-8/3}} + \cancel{4/3te^{-8/3t}} =$$

$$e^{-8/3t}$$ ✓

defined for all t, fundamental set

4. Use the method of reduction of order and the stated solution to find a second solution to the second order equation $t^2y'' + 2ty' - 2y = 0, t > 0, y_1(t) = t$. (20 points)

$y_2(t) = v(t)\,t$

$y_2'(t) = v't + v$

$y_2''(t) = v''t + v' + v' = v''t + 2v'$

$t^2(v''t + 2v') + 2t(v't + v) - 2(vt) = 0$

$v''t^3 + v'(2t^2 + 2t^2) + v(\cancel{2t - 2t}) = 0$

$\dfrac{v''t^3 + v'(4t^2) = 0}{t^2}$

let $u = v'$

$u' = v''$

$v''t + v'4 = 0$

$u't + 4u = 0 \Rightarrow \dfrac{u'\cancel{t}}{u\cancel{t}} = -\dfrac{4\cancel{u}}{\cancel{u}t} \Rightarrow \int \dfrac{du}{u} = \int -\dfrac{4\,dt}{t}$

$\ln u = -4\ln t \Rightarrow \ln u = \ln t^{-4} \Rightarrow u = t^{-4}$

$u = v' \Rightarrow v = \int t^{-4}\,dt = -\frac{1}{3}t^{-3}$

$\boxed{y_2 = t^{-3}t = 1/t^3 \cdot t = \frac{1}{t^2}}$

5. For each of the systems below, find the general solution to the forcing term. You should use the method of undetermined coefficients on at least one problem, and variation of parameters on at least one problem (the third is your choice). [Note: you found the solutions to the homogeneous case in Problem #1.] (15 points each)

a. $y''+8y'-9y=e^{-t}+1, y(1)=1, y'(1)=0$

$y_1 = e^{-9t}$ $\quad y_2 = e^{t}$ $\quad Y_1(t) = Ce^{-t}$ $\quad Y_1'(t) = -Ce^{-t}$ $\quad Y_1''(t) = Ce^{-t}$

$Ce^{-t} - 8Ce^{-t} - 9Ce^{-t} = e^{-t}$ $\qquad Y_2(t) = D$

$C - 8C - 9C = 1$ $\qquad -9D = 1 \quad D = -1/9$

$-16C = 1$

$C = -1/16$

$$\boxed{y(t) = .0533e^{-9t} + .4172e^{t} - 1/16\, e^{-t} - 1/9}$$

$y(t) = Ae^{-9t} + Be^{t} - 1/16\, e^{-t} - 1/9$ $\qquad y'(t) = -9Ae^{-9t} + Be^{t} + 1/16\, e^{-t}$

$1 = Ae^{-9} + Be - 1/16\, e^{-1} - 1/9$ $\qquad 0 = -9A + B + 1/16$

$10/9 + 1/16e = Ae^{-9} + Be$ $\qquad -1/16 = -9A + B$

$A = \dfrac{160e^{9} + 9e^{8} - 1}{144(9e^{10}+1)} + \dfrac{1}{144} \approx .0533$ $\qquad B = \dfrac{160e^{9} + 9e^{8} - 1}{16(9e^{10}+1)} \approx .4172$

error on undetermined coefficients only. *

b. $y''-2y'+4y=-t^3, y(0)=1, y'(0)=0$

$y_1 = e^{t}\cos\sqrt{3}t$ $\quad y_2 = e^{t}\sin\sqrt{3}t$ $\qquad Y(t) = At^3 + Bt^2 + Ct + D$

$Y_1'(t) = 3At^2 + 2Bt + C$

$Y''(t) = 6At + 2B$

$6At + 2B - 2(3At^2 + 2Bt + C) + 4(At^3 + Bt^2 + Ct + D) = -t^3$

$4At^3 = -t^3 \Rightarrow 4A = -1 \Rightarrow A = -1/4$

$-6At^2 + 4Bt^2 = 0 \Rightarrow 4B = -6/4 = -3/2 \Rightarrow B = -3/8$

$6At - 4Bt + 4Ct = 0 \Rightarrow 6A - 4B + 4C = 0 \Rightarrow -6/4 - 3/2 + 4C = 0$

$-3/2 - 3/2 = -3$

$4C = 3 \Rightarrow C = 3/4$

$2B - 2C + 4D = 0 \Rightarrow -3/4 - 6/4 + 4D = 0 \Rightarrow 4D = 9/4 \Rightarrow D = 9/16$

$Y(t) = -1/4t^3 - 3/8t^2 + 3/4t + 9/16$

$y(t) = Ae^{t}\cos\sqrt{3}t + Be^{t}\sin\sqrt{3}t - 1/4t^3 - 3/8t^2 + 3/4t + 9/16$

$1 = A + 9/16$

$7/16 = A$

$y'(t) = -\sqrt{3}A\sin\sqrt{3}t + Ae^{t}\cos\sqrt{3}t + Be^{t}\sin\sqrt{3}t + \sqrt{3}Be^{t}\cos\sqrt{3}t - 3/4t^2 - 3/4t + 3/4$

$0 = A + \sqrt{3}B + 3/4 \Rightarrow \sqrt{3}B = -19/16 \quad B = \dfrac{-19}{16\sqrt{3}}$

$$\boxed{y(t) = 7/16\, e^{t}\cos\sqrt{3}t - \tfrac{19}{16\sqrt{3}}\cos\sqrt{3}t - 1/4t^3 - 3/8t^2 + 3/4t + 9/16}$$

5a. $Y(t) = -y_1(t)\int \frac{y_2(t)\,g(t)}{W}\,dt + y_2(t)\int \frac{y_1(t)\,g(t)}{W}\,dt$

$$-e^{-9t}\int \frac{e^{t}(e^{-t}+1)}{10e^{-8t}}\,dt + e^{t}\int \frac{e^{-9t}(e^{-t}+1)}{10e^{-8t}}\,dt$$

$$= -e^{-9t}\int \frac{1+e^{t}}{10e^{-8t}}\,dt + e^{t}\int \frac{e^{-t}(e^{-t}+1)}{10}\,dt$$

$$= -\frac{1}{10}e^{-9t}\int e^{8t}+e^{9t} + \frac{1}{10}e^{t}\int e^{-2t}+e^{-t}\,dt$$

$$= -\frac{1}{10}e^{-9t}\left[\frac{1}{8}e^{8t}+\frac{1}{9}e^{9t}\right] + \frac{1}{10}e^{t}\left[-\frac{1}{2}e^{-2t}-e^{-t}\right] =$$

$$= -\frac{1}{80}e^{-t} - \frac{1}{90}\cancel{e^{0}} + -\frac{1}{20}e^{-t} - \frac{1}{10}\cancel{e^{0}} =$$

$$-\tfrac{1}{16}e^{-t} - \tfrac{1}{9}$$

Solve the rest as w/ undetermined coeff.

5b. $Y(t) = -e^{t}\cos\sqrt{3}t\int \frac{e^{t}\sin\sqrt{3}t\cdot(-t^{3})}{\sqrt{3}e^{2t}} + e^{t}\sin\sqrt{3}t\int \frac{e^{t}\cos\sqrt{3}t\,(-t^{3})}{\sqrt{3}e^{2t}}$

I strongly recommend using undetermined coefficients rather than fussing w/ triple by parts.

$$-\frac{\cancel{e^{t}}}{\sqrt{3}}\cos\sqrt{3}t\left[\frac{\cancel{e^{-t}}}{16}\left(4\cancel{\sqrt{3}}t^{3}\cos\sqrt{3}t\right) + 6\cancel{\sqrt{3}}t^{2}\cos\sqrt{3}t - 3\cancel{\sqrt{3}}\cos\sqrt{3}t\right) + \cancel{\left(4t^{3}-6t^{2}-12t-3\right)\sin\sqrt{3}t}\,)\Big] +$$

cancels each other

$$\frac{\cancel{e^{t}}}{\cancel{\sqrt{3}}}\sin\sqrt{3}t\left[\frac{\cancel{e^{-t}}\cancel{\sqrt{3}}}{16}\left(4t^{3}\cancel{\cos\sqrt{3}t) - 6t^{2}\cos\sqrt{3}t - 12t\cos\sqrt{3}t - 3\cos\sqrt{3}t}\right) - \cancel{\sqrt{3}\sin\sqrt{3}t\,(4t^{3}+6t^{2}-3)}\right] =$$

$$= \left(\tfrac{1}{4}t^{3} - \tfrac{3}{8}t^{2} + \tfrac{3}{16}\right)\underbrace{\left(\cos^{2}\sqrt{3}t + \sin^{2}\sqrt{3}t\right)}_{=1}$$

* These are correct

5c. $Y(t) = -e^{-4/3t}\int \frac{te^{-4/3t}6\sin 2t}{e^{-8/3t}}\,dt + te^{-4/3t}\int \frac{e^{-4/3t}6\sin 2t}{e^{-8/3t}}\,dt$

$= e^{-4/3}\int te^{4/3t}\,6\sin 2t\,dt + te^{-4/3t}\int e^{4/3t}\,6\sin 2t\,dt$

$= e^{-4/3t}\left[\frac{-9e^{+4/3t}}{338}(78t\cos 2t \mp 36\cos 2t \mp (52t+15)\sin 2t)\right]$

$+ te^{-4/3t}\left[\frac{-9e^{4/3t}}{13}(3\cos 2t - 2\sin 2t)\right]$

$= -\frac{18}{169}\cos 2t - \frac{9(\cancel{52t}+15)\sin 2t}{338} + \cancel{\frac{18t\sin 2t}{13}}$

$= \frac{-18}{169}\cos 2t - \frac{15}{338}\sin 2t$

probably easier w/ undetermined coefficients

c. $9y'' + 24y' + 16y = 6\sin 2t,\ y(0) = 2,\ y'(0) = -1$

$y_1 = e^{-4/3 t}$ $\quad y_2 = te^{-4/3 t}$ $\quad Y(t) = A\sin 2t + B\cos 2t$

$Y'(t) = 2A\cos 2t - 2B\sin 2t$

$Y''(t) = -4A\sin 2t - 4B\cos 4t$

$9(-4A\sin 2t - 4B\cos 4t) + 24(2A\cos 2t - 2B\sin 2t) + 16(A\sin 2t + B\cos 2t) = 6\sin 2t$

$\sin 2t: \ -36A - 48B + 16A = 6 \Rightarrow -20A - 48B = 6$

$\cos 2t: \ -36B + 48A + 16B = 0 \qquad 48A - 20B = 0$

$A = \frac{-15}{338} \quad B = \frac{-18}{169} \qquad Y(t) = \frac{-15}{338}\sin 2t - \frac{18}{169}\cos 2t$

$y(t) = Ae^{-4/3t} + Bte^{-4/3t} - \frac{15}{338}\sin 2t - \frac{18}{169}\cos 2t$

$2 = A - \frac{18}{169} \Rightarrow A = \frac{356}{169}$

$y'(t) = -4/3\, Ae^{-4/3t} + Be^{-4/3t} - 4/3\, Bte^{-4/3t} - \frac{15}{169}\cos 2t + \frac{36}{169}\sin 2t$

$-1 = -4/3\left(\frac{356}{169}\right) + B - \frac{15}{169} \Rightarrow B = 74/39$

$$\boxed{y(t) = \frac{356}{169}e^{-4/3t} + 74/39\, te^{-4/3t} - \frac{15}{338}\sin 2t - \frac{18}{169}\cos 2t}$$

6. A series circuit has a capacitor of 0.5×10^{-5} F and an inductor of 0.1 H. If the initial charge on the capacitor is 10^{-6} C and there is no initial current, find the charge Q on the capacitor at any time t. What is the current at any time t? (15 points)

$LQ'' + RQ' + \frac{1}{C}Q = 0 \qquad Q(0) = 10^{-6}$

$\qquad Q'(0) = 0$

$0.1Q'' + \frac{1}{.5 \times 10^{-5}}Q = 0$

$.1r^2 + \frac{1}{.5 \times 10^{-5}} = 0$

$r^2 = \left(\frac{-1}{.5 \times 10^{-5}}\right)/.1 = -2 \times 10^6 \Rightarrow r = \sqrt{2}i \times 10^3$

$Q(t) = A\cos(\sqrt{2} \times 10^3)t + B\sin(\sqrt{2} \times 10^3)t$

$Q'(t) = -\sqrt{2} \times 10^3 A\sin(\sqrt{2} \times 10^3 t) + (\sqrt{2} \times 10^3)B\cos(\sqrt{2} \times 10^3)t$

$10^{-6} = A$

$$\boxed{Q(t) = 10^{-6}\cos(\sqrt{2} \times 10^3 t)}$$

$0 = \sqrt{2} \times 10^3 B \Rightarrow B = 0 \qquad I(t) = -\sqrt{2} \times 10^{-3}\sin(\sqrt{2} \times 10^3 t)$

$=.1m$

7. A mass of 5 kg stretches a spring 10 cm. The mass is acted on by an external force of $10\sin\left(\frac{t}{2}\right)$ N and moved in a medium that imparts a viscous force of 2 N when the speed of the mass is 4 cm/sec. (35 points)

   a. If the mass is set in motion from its equilibrium position with an initial velocity of 3 cm/sec, formulate the initial value problem describing the motion of the mass.

$F_d = 2 = \gamma(.04) \Rightarrow \gamma = 50$ $\quad y(0)=0 \quad y'(0)=.03$

$F_s = \underset{\text{force} = ma}{\underline{5\times 9.8}} = k(.1) \quad k=490$

$5y'' + 50y' + 490y = 10\sin(t/2)$

   b. Solve the equation you found in part a.

$5r^2 + 50r + 490 = 0$ $\quad r = \dfrac{-10 \pm \sqrt{100-392}}{2} = -5 \pm \dfrac{\sqrt{292}}{2} =$

$r^2 + 10r + 98 = 0$ $\quad -5 \pm \sqrt{73}$

$y(t) = e^{-5t}(A\cos\sqrt{73}t + B\sin\sqrt{73}t)$

$Y(t) = C\sin(\tfrac{1}{2}t) + D\cos\tfrac{t}{2}$ $\quad Y'(t) = \frac{C}{2}\cos\frac{t}{2} - \frac{D}{2}\sin\frac{t}{2}$ $\quad Y''(t) = -\frac{C}{4}\sin\frac{t}{2} - \frac{D}{4}\cos\frac{t}{2}$

$-\frac{5C}{4}\sin\frac{t}{2} - \frac{5D}{4}\cos\frac{t}{2} + 25C\cos\frac{t}{2} - 25D\sin\frac{t}{2} + 490C\sin\frac{t}{2} + 490D\cos\frac{t}{2} = 10\sin\frac{t}{2}$

$\sin\frac{t}{2}: \; -\frac{5C}{4} - 25D + 490C = 10 \Rightarrow \frac{1955}{4}C - 25D = 10$

$\cos\frac{t}{2}: \; -\frac{5D}{4} + 25C + 490D = 0 \quad\quad 25C + \frac{1955D}{4} = 0$

$C = \frac{3128}{153281}, \quad D = \frac{-160}{153281}$

$\approx .0204 \quad\quad \approx .00104$

$y(t) = e^{-5t}(A\cos\sqrt{73}t + B\sin\sqrt{73}t) + \frac{3128}{153281}\sin\frac{t}{2} - \frac{160}{153281}\cos\frac{t}{2}$

$y'(t) = -5e^{-5t}(A\cos\sqrt{73}t + B\sin\sqrt{73}t) + e^{-5t}(\sqrt{73}A\sin\sqrt{73}t + \sqrt{73}B\cos\sqrt{73}t) + \frac{1564}{153281}\cos\frac{t}{2} + \frac{80}{153281}\sin\frac{t}{2}$

$0 = A - \frac{160}{153281} \Rightarrow A = \frac{160}{153281}$

$.03 = -5A - \sqrt{73}B + \frac{1564}{153281} \quad\quad B = -.001706$

   c. Describe the damping of the system: undamped, underdamped, critically damped or overdamped.

underdamped

$\boxed{y(t) = .00104e^{-5t}\cos\sqrt{73}t - .001706e^{-5t}\sin\sqrt{73}t + .0204\sin\frac{t}{2} - .00104\cos\frac{t}{2}}$

d. What is the transient solution and what is the steady state solution?

transient : $.00104e^{-5t}\cos\sqrt{73}t - .001706e^{-5t}\sin\sqrt{73}t$

Steady state : $.0204\sin t/2 - .00104\cos t/2$

e. Does the solution achieve resonance or does it contain beats? (Or neither.)

it will contain beats since there are 2 pairs of trig functions, but they will quickly be unnoticeable

f. How does the natural frequency (without damping) compare to the quasi-frequency?

w/o damping

$5r^2 + 490 = 0$ natural frequency is faster

$r^2 + 98 = 0$

$r = \pm\sqrt{98}\,i \approx 9.89949\ldots$ vs. $\sqrt{73} \approx 8.544$

g. What is the long-term behaviour of the system as t→∞?

it rapidly approaches the steady state solution and oscillates w/ the driving force

h. Write the solution to the system (the homogeneous portion only) as $y = R\cos(\omega t - \delta)$ and clearly state the amplitude, the period (or quasi-period), the phase shift.

$y = e^{-5t}(A\cancel{\cos\sqrt{73}t} + B\sin\sqrt{73}t) \Rightarrow y' = -5e^{-5t}B\sin\sqrt{73}t + e^{-5t}B\sqrt{73}\cos\sqrt{73}t$

w/o driver: $.3 = \cancel{-50} + B\sqrt{73}$ $B = \frac{.3}{\sqrt{73}}$

$0 = A$

$y(t) = \frac{.3}{\sqrt{73}}\sin\sqrt{73}t$

amplitude $= \frac{.3}{\sqrt{73}}$ quasi-period $= \frac{2\pi}{\sqrt{73}}$ phase shift $\pi/2$

since there is only a sine term

i. State the values of the first 4 times y=0 (if fewer than that, state them all for t>0).

$t_1 = 6.38545$ $\approx 2\pi$

$t_2 = 12.6686$ $\approx 4\pi$

$t_3 = 18.9518$ $\approx 6\pi$

$t_4 = 25.235$ $\approx 8\pi$