Math 268, Homework #6, Spring 2012 Name ____

Instructions: Write your work up neatly and attach to this page. Record your final answers (only) *directly on this page*. Use exact values unless specifically asked to round.

1. In a certain forest, a feral cat colony preys on chipmunks according to the predator-prey model given by $A = \begin{bmatrix} .35 & .25 \\ -p & 1.25 \end{bmatrix}$.
   a. Suppose that the predation parameter p is given by 0.35, 0.5 and 0.7 respectively. Determine the long-term behavior in each case. (What is the ratio of cats to chipmunks in the long run?)
   b. For each of the cases above, find the eigenvalues and associated eigenvectors of the matrix and plot a trajectory for each value of p starting from the initial condition $\vec{x}_0 = \begin{bmatrix} 10 \\ 15 \end{bmatrix}$, $\vec{y}_0 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and $\vec{z}_0 = \begin{bmatrix} 2 \\ 10 \end{bmatrix}$
   c. Describe the behavior of the origin for each value of p. Is the origin an attractor, a repeller or a saddle point?

2. For each pair of vectors in i-iii, find the following:
   a. $\vec{u} \cdot \vec{v}$
   b. $||\vec{u}||$ and $||\vec{v}||$
   c. $\frac{\vec{u}}{||\vec{u}||}$ and $\frac{\vec{v}}{||\vec{v}||}$
   d. $||\vec{u}||^2 + ||\vec{v}||^2$
   e. $||\vec{u} + \vec{v}||^2$
   f. $\frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v}$
   g. $||\vec{u} - \vec{v}||$

   i. $\vec{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

   ii. $\vec{u} = \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix}, \vec{v} = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$

   iii. $\vec{u} = \begin{bmatrix} 3 \\ 2 \\ -5 \\ 0 \end{bmatrix}, \vec{v} = \begin{bmatrix} -4 \\ 1 \\ -2 \\ 6 \end{bmatrix}$

3. Use the information in problem #2, for each pair of vectors in i-iii, to determine the following:
   a. The distance between $\vec{u}$ *and* $\vec{v}$.
   b. Are the vectors $\vec{u}$ *and* $\vec{v}$ orthogonal?
   c. What is the angle between $\vec{u}$ *and* $\vec{v}$?
   d. Find unit vectors in the direction of $\vec{u}$ *and* $\vec{v}$.
   e. The orthogonal projection of $\vec{u}$ in the direction of $\vec{v}$.
   f. If $\vec{u}$ *and* $\vec{v}$ are orthogonal, call the subspace spanned by the vectors W and find an orthonormal basis for the subspace.
   g. Find $W^{\perp}$.

4. Show that the vectors $\vec{u}_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$ form an orthogonal basis for $\mathbb{R}^3$. Make this basis an orthonormal basis, and then use that basis to find the representation of $\vec{x} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$ in that basis using the formula $\vec{x} = c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 \; where \; c_j = \frac{\vec{x}\cdot\vec{u}_j}{\vec{u}_j\cdot\vec{u}_j} \; (j = 1,2,3)$.

5. Separate $\vec{y} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ into $\vec{y}_{\parallel} \; and \; \vec{y}_{\perp}$ if $\vec{y}_{\parallel}$ is in the direction of $\vec{u} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$.

6. For each statement, indicate whether it's true or false. For the ones that are false, state the correct true statement.
   a. Not every linearly independent set in $\mathbb{R}^n$ is an orthogonal set.

   b. If $\vec{y}$ is a linear combination of nonzero vectors from an orthogonal set, then the weights in the linear combination can be computed without row operations on a matrix.

   c. If the vectors in an orthogonal set are normalized, then some of the new vectors may not be orthogonal.

   d. If the columns of an mxn matrix A are orthonormal, then the linear mapping $\vec{x} \mapsto A\vec{x}$ preserves lengths.

   e. An orthogonal matrix is invertible.

   f. If $\vec{x}$ is orthogonal to $\vec{u}_1 \; and \; \vec{u}_2$ and if W = Span{**u**$_1$, **u**$_2$}, then $\vec{x}$ must be in $W^{\perp}$.

   g. If $\vec{y}$ is in a subspace W, then the orthogonal projection of $\vec{y}$ onto W is $\vec{y}$ itself.

   h. If the columns of an nxp matrix U are orthonormal, then $UU^T\vec{y}$ is the orthogonal projection of $\vec{y}$ onto the column space of U.

   i. In the Orthogonal Decomposition Theorem, each term in the formula $\vec{y}_{\parallel} = \sum_{i=1}^{p} \frac{\vec{y}\cdot\vec{u}_i}{\vec{u}_i\cdot\vec{u}_i}$ is itself an orthogonal projection of $\vec{y}$ onto a subspace of W.

   j. The best approximation to $\vec{y}$ by elements of a subspace W is given by the vector $\vec{y} - proj_W\vec{y}$.

   k. The general least-squares problem is to find an $\vec{x}$ that makes $A\vec{x}$ as close to $\vec{b}$ as possible.

   l. Any solution of $A^TA\vec{x} = A^T\vec{b}$ is a least-squares solution of $A\vec{x} = \vec{b}$.

   m. If the columns of A are linearly independent, then the equation $A\vec{x} = \vec{b}$ has exactly one least-squares solution.

   n. A least-squares solution of $A\vec{x} = \vec{b}$ is the point in the column space of A closest to $\vec{b}$.

7. Verify that the given set of vectors {u1,...,un} is orthonormal, and then write $\vec{x}$ as a pair or vectors $\overrightarrow{x_\parallel}$ *and* $\overrightarrow{x_\perp}$, with W defined as the span of the specified vectors and $\overrightarrow{x_\parallel}$ in W. What is the best approximation to $\vec{x}$ in W? What is the distance from the subspace to the point $\vec{x}$.

   a. $\vec{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix}, \vec{u}_4 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}, W = Span\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}, \vec{x} = \begin{bmatrix} 4 \\ 5 \\ -3 \\ 3 \end{bmatrix}$

   b. $\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, W = Span\{\vec{u}_1, \vec{u}_2\}, \vec{x} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$

   c. $\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}, W = Span\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}, \vec{x} = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$

8. Find the least-squares approximation for $A\vec{x} = \vec{b}$.

   a. $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 3 \\ 8 \\ 2 \end{bmatrix}$

   b. $A = \begin{bmatrix} 2 & 1 \\ -2 & 0 \\ 2 & 3 \end{bmatrix}, \vec{b} = \begin{bmatrix} -5 \\ 8 \\ 1 \end{bmatrix}$

9. Find the best-fit equation specified for the given set of data.

   a. $\{(1,0), (2,1), (4,2), (5,3)\}, y = \beta_0 + \beta_1 x$

   b. $\{(1,0), (2,1), (4,2), (5,3)\}, y = \beta_0 + \beta_1 x + \beta_2 x^2$

   c. $\{(4,1.58), (6,2.08), (8,2.5), (10,2.8), (12,3.1), (14,3.4), (16,3.8), (18,4.32)\}, y = \beta_1 x + \beta_2 x^2 + \beta_3 x^3$