Betsy McCall

Solving Trigonometric Equations

In order to solve trigonometric equations (equations involving trigonometric functions), we will largely be trying to reduce these equations to familiar techniques in order to isolate single trig functions. Once the trig functions are isolated, then we can employ inverse trig functions to finish off the solution.

Let's begin with a simple linear case.

1. **Linear Trigonometric Equation, θ, one trig function**

   $2\sin\theta + 3 = 2$

   In this linear equation, for the moment, pretend that $\sin\theta$ is replaced by x. Follow the same steps that you would follow to solve 2x+3 = 2.

   $$\begin{aligned} 2x + 3 &= 2 \\ -3 \ &\ -3 \\ \hline 2x &= -1 \\ \overline{2} \ &\ \overline{2} \\ x &= -\frac{1}{2} \end{aligned} \qquad \begin{aligned} 2\sin\theta + 3 &= 2 \\ -3 \ &\ -3 \\ \hline 2\sin\theta &= -1 \\ \overline{2} \ &\ \overline{2} \\ \sin\theta &= -\frac{1}{2} \end{aligned}$$

   At this point, we need to find values of θ. In typical problems, we will be finding values of θ between 0 and 2π (i.e. one turn around the unit circle). However, our inverse trig functions will not get us both the values (in most cases, there will be two), and for four of the trig functions, the value we get from the inverse trig function may not even originally be in this range (they will be negative angles in the fourth quadrant). For this reason, for special angles, we are better off going by memory, than by using the calculator. (We will talk about the calculator issue later).

   Here, $\sin\theta = -\frac{1}{2}$ or $\sin^{-1}\left(-\frac{1}{2}\right) = \theta$, means that we are looking for an angle for which the sine of that angle is -1/2. If we use reference angles and the 30-60-90 triangle, we come up with $-\frac{\pi}{6}$. This angle is in the fourth quadrant, but not in the 0 to 2π range. We can add 2π to this value to get it in the correct range. Our angle is now $\frac{11\pi}{6}$. The second angle is in the third quadrant. Add our original angle to π to the reference angle get $\frac{7\pi}{6}$. Thus, our solution set is $\left\{\frac{7\pi}{6}, \frac{11\pi}{6}\right\}$. If we needed more values, we can add 2kπ.

   Calculator Note: We can get the same result from the calculator. Ensure that your calculator is in radian mode. Type in $\sin^{-1}\left(-\frac{1}{2}\right)$ *to get = -.523598…. To get this in fractional form, divide by π, and convert the remainder to a fraction. The rest of the process is the same.*

2. **Linear Trigonometric Equation, multiple of θ, one trig function**

$\tan(2\theta) = -1$

A problem like this is going to be solved the same way as the problem above, but with two differences. First, we will solve for 2 θ as a unit, or as if we set 2 θ =α, and solved for α in $\tan(\alpha) = -1$. The second difference is that for each multiple of θ, we will need to find that many more multiple solutions. For example, for most equations, θ will have two solutions. If our angle now is 2θ, we will need solutions; if it's 3θ, we will need six solutions. We can find these values for α but going around the circle sufficient number of times to acquire these values, one more time for each pair needed. Then solve for θ.

In our example, $\tan(\alpha) = -1$ or $\tan^{-1}(-1) = \alpha$. The principle solution provided by the inverse trig function is $-\frac{\pi}{4}$. To get this into the range of 0 to 2π we add π: once to get an angle in the 2nd quadrant, and once more to get a positive angle in the fourth quadrant. If we were just solving for $\tan(\theta) = -1$, then we would report our two solutions $\left\{\frac{3\pi}{4}, \frac{7\pi}{4}\right\}$ and be finished. But we still have to deal with that 2θ instead. Therefore, we need a total of **four** *solutions instead of two*. So, add π again, twice more to get two more values. $\left\{\frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4}\right\}$ as values for α, but α=2θ. Our four values for θ, therefore, are $\left\{\frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}\right\}$. All these values are between 0 and 2π, and you can check for yourself, that all these values will solve the original equation.

3. **Linear Trigonometric Equation, linear equation in θ, one trig function**

$\cos\left(\frac{\theta}{3} - \frac{\pi}{4}\right) = \frac{1}{2}$

As with example #2 above, we are at first going to ignore the crazy angle expression inside the cosine function and replace it with α: $\cos(\alpha) = \frac{1}{2}$. With our multiplier of 1/3, we may only end up keeping one value, but with the π/4 in the problem, it's not going to be clear at first what value is going to work, or indeed, if we may still have two values, so it will be best to find a couple values and then eliminate whatever we don't need, or that fall outside the desired range. Possible values for α then are $\frac{\pi}{3}$, and let's pick one smaller one $-\frac{\pi}{3}$ and one bigger $\frac{2\pi}{3}$. Since all these are possible values for α, we have three equations for θ:

$$\frac{\theta}{3}-\frac{\pi}{4}=\frac{\pi}{3}, \qquad \frac{\theta}{3}-\frac{\pi}{4}=-\frac{\pi}{3}, \qquad \frac{\theta}{3}-\frac{\pi}{4}=\frac{2\pi}{3}$$

$$+\frac{\pi}{4}\quad\frac{\pi}{4} \qquad +\frac{\pi}{4}\quad\frac{\pi}{4} \qquad +\frac{\pi}{4}\quad\frac{\pi}{4}$$

$$\frac{\theta}{3}=\frac{7\pi}{12} \qquad \frac{\theta}{3}=-\frac{\pi}{12} \qquad \frac{\theta}{3}=\frac{11\pi}{12}$$

$$\theta=\frac{7\pi}{4} \qquad \theta=-\frac{\pi}{4} \qquad \theta=\frac{11\pi}{4}$$

We can see from these possible values that only the first one is within 0 and 2π, so that one gets kept. Our solution set then is $\left\{\frac{7\pi}{4}\right\}$.

4. Quadratic Trigonometric Function, θ, no linear term

$4\cos^2\theta-3=0$

Like any quadratic equation of this type, for instance, $4x^2 - 3 = 0$, we will solve for the squared-term and then take the square root of both sides. This will serve to double the number of solutions we would expect for a linear equation. An example like this will typically have four solutions.

$$4x^2-3=0 \qquad 4\cos^2\theta-3=0$$

$$4x^2=3 \qquad 4\cos^2\theta=3$$

$$x^2=\frac{3}{4} \qquad \cos^2\theta=\frac{3}{4}$$

$$x=\pm\sqrt{\frac{3}{4}}=\pm\frac{\sqrt{3}}{2} \qquad \cos\theta=\pm\sqrt{\frac{3}{4}}=\pm\frac{\sqrt{3}}{2}$$

So, we then have to find all angle values such that $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)=\theta$ and $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)=\theta$.

This is going to before four angles, and here, all the angles with $\frac{\pi}{6}$ as the reference angle: $\left\{\frac{\pi}{6},\frac{5\pi}{6},\frac{7\pi}{6},\frac{11\pi}{6}\right\}$.

5. Quadratic Trigonometric Equation, θ, linear term present

$2\cos^2\theta+\cos\theta-1=0$

Here, we are going to treat these like equations in quadratic form, i.e. we will make a substitution for the common trig function so that the equation is quadratic. The solutions to the quadratic will be possible values for the trig function, for which we will then have to solve. Note that since the quadratic equation can give us two solutions, and the trig function can have two solutions, we may end up with as many as four solutions in the 0 to 2π range.

$$2\cos^2\theta + \cos\theta - 1 = 0$$
$$let\ \cos\theta = x$$
$$2x^2 + x - 1 = 0$$
$$(2x-1)(x+1) = 0$$
$$x = \frac{1}{2}, x = -1 \therefore \cos\theta = \frac{1}{2}, \cos\theta = -1$$

Angles that give us solutions to the first values of cosine are $\left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}$. There is only one angle for sine or cosine equal to one or negative one, so the second value for cosine gives only $\{\pi\}$. Thus, our three solutions to this equation are: $\left\{\frac{\pi}{3}, \pi, \frac{5\pi}{3}\right\}$.

6. **Quadratic Trigonometric Equation, θ, mixed trig functions**

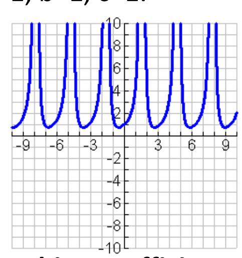
$$\sec^2\theta + \tan\theta = 0$$

To solve an equation with mixed trig functions, we will want to try to reduce them to a common trig function, if possible. In many cases, the trig functions will be related to each other through an identity. In this example, we have the Pythagorean identity $\sec^2\theta = 1 + \tan^2\theta$. We will use this identity to replace the secant function in the equation. It should be noted that we can swap out the squared-terms with the Pythagorean identities more easily than the linear terms. Therefore, it is the linear term that we wish to *match* through our substitutions with identities. Making the replacement here, our equation becomes:

$$\tan^2\theta + \tan\theta + 1 = 0$$

Recall, as with the above problem, we will make a substitution, either explicitly or mentally, letting $x = \tan\vartheta$ here, and solving the equations $x^2+x+1=0$. If the equation is factorable, then we should factor, however, if not, we can use the quadratic formula, with a=1, b=1, c=1.

$$\tan\theta = x = \frac{-1 \pm \sqrt{1-4(1)(1)}}{2(1)} = \frac{-1 \pm \sqrt{-3}}{2}$$



This particular solution is imaginary, and so there are no real solutions to this particular equation. We can verify this by graphing the function to see that there are no intercepts with the x-axis.

7. **Linear Trigonometric Equation, matching θ, mixed trig functions and matching coefficients**

$$\sin\theta + \cos\theta = \sqrt{2}$$

Unlike in previous examples, we have two linear terms with different trig functions. If the constant here was equal to zero, we'd be able to just ask ourselves when were the two functions the same or different only by sign. We don't have that option here. Our technique in this case, then, is to square both sides of the equation. This will give us squared terms and we'll be able to apply identities to our problem.

$$(\sin\theta+\cos\theta)^2=\left(\sqrt{2}\right)^2$$
$$\sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta=2$$

Since we know that $\sin^2\theta+\cos^2\theta=1$, we can reduce this equation to just a constant and the middle term.

$$2\sin\theta\cos\theta+1=2$$
$$2\sin\theta\cos\theta=1$$

Solving this problem then requires another trigonometric identity: the double angle formula for sine.

$$2\sin\theta\cos\theta=\sin(2\theta)$$
$$\sin(2\theta)=1$$

The problem now has been reduced to example #2. Our α values are $\left\{\frac{\pi}{2},\frac{5\pi}{2}\right\}$. (Recall, that we need values between 0 and 4π for α because α=2θ, and once we divide by 2, this will ensure our values are in the desired range when we are finished. So, our possible θ values are $\left\{\frac{\pi}{4},\frac{5\pi}{4}\right\}$. However, we are not yet done. Because we squared the original problem, we are actually solving for values that satisfy both $\sin\theta+\cos\theta=\sqrt{2}$ and $\sin\theta+\cos\theta=-\sqrt{2}$. We will have to check our solutions, just as with a radical problem, in order to eliminate extraneous solutions. As it turns out, only $\left\{\frac{\pi}{4}\right\}$ works for the positive square root.

Note: If you encounter problems with non-identical coefficients, there is a means of solving them by hand, but it's not pretty. For our purposes, solve these in your calculator, and we'll cover that below.

8. Miscellaneous Trigonometric Equations, mismatched θ, mismatched trig functions

This is the catch-all category where things get trickier, and it's all but impossible to list all the conceivable scenarios you might encounter. All of these problems, though, ultimately reduce to examples we've covered already.

a. $\cos(2\theta)+6\sin^2\theta=4$

When θ doesn't match from term to term, our first goal is to get all the angles to be the same, or we have no hope of solving by hand. Cos(2θ) has several identity options for replacing it in this equation, and we want to choose the one that with make the problem easiest to simplify. Therefore, in this problem we want the identity $\cos(2\theta)=1-2\sin^2\theta$. This problem then reduces to example #4.

b. $\sin(2\theta)\sin\theta=\cos\theta$

This one is a bit different, though we do still need to apply the double angle formula, this time for sine. Our problem then reduces, and put all terms on one side.

$2\sin^2\theta\cos\theta-\cos\theta=0$

In this version of the problem, we don't have to try to convert to all the same trig function because the cosine function can be factored out of both terms.

$\cos\theta\left(2\sin^2\theta-1\right)=0$

From here, we can set each factor separately equal to zero. We will get potentially six solutions.

c. $\sin(2\theta)+\sin(4\theta)=0$ or $\sin(4\theta)-\sin(6\theta)=0$

Problems like these need to be converted to equivalent equations with matching angles. They need not be converted all the way down to θ. In the first example, applying the double angle formula to the second term gives:

$\sin(2\theta)+2\sin(2\theta)\cos(2\theta)=0$, and as with b above we can factor it from here.

In the second example, this is much harder, but try using the sum and different formulas by letting 6θ=2θ+4θ. If that doesn't reduce the problem to something you can work with, continue with the double angle formulas to reduce the 4θs to 2θs.

d. $\sec\theta=\tan\theta+\cot\theta$

The best strategy here is to convert everything to sine and cosine and eliminate the denominators. That will reduce the problem to a previous case.

9. Mixed Trigonometric and Algebraic Equations

Sometimes problems can't be solved by hand. When you mix naked variables with trig functions, or with exponentials, these cannot be solved by hand with traditional techniques. They can only be solved numerically, and for us, that means, turning to the calculator. Examples would be $x-\sin x=0$ or $4\cos(3x)-e^x=1$. The number of solutions is unpredictable, and if the number of solutions is finite, you should find them all, even if they are outside the range of 0 to 2π. If the number is infinite, you can stick with that range. Remember that your calculator should be in radian mode. Use the zero function or the intersect mode (depending on how you enter the equation into your calculator). Report answers with at least two decimal places accuracy. Do no attempt to estimate the solutions with the Trace feature.

Problems.

i. $2\sin\theta+1=0$

ii. $4\sec\theta+6=-2$

iii. $\tan\theta+1=0$

iv. $\sin(3\theta)=-1$

v. $\cot\left(\frac{2\theta}{3}\right)=-\sqrt{3}$

vi. $\cos\left(2\theta - \frac{\pi}{2}\right) = -1$

vii. $\sin\left(3\theta + \frac{\pi}{18}\right) = 1$

viii. $\tan(2\theta) = -1$

ix. $\tan\theta = 5$

x. $\csc\theta = -3$

xi. $\sin^2\theta - 1 = 0$

xii. $(\cot\theta + 1)(2\csc\theta - 1) = 0$

xiii. $4\cos^2\theta - 3 = 0$

xiv. $\tan^2\theta = 3$

xv. $\sin^2\theta = 6(\cos\theta + 1)$

xvi. $\cos(2\theta) = 2 - 2\sin^2\theta$

xvii. $\tan\theta = \sin\theta$

xviii. $\sin(2\theta) = \cos\theta$

xix. $\sin(2\theta) - \cos(4\theta) = 0$

xx. $\csc^2\theta = \cot\theta + 1$

xxi. $2\cos^2\theta - 7\cos\theta - 4 = 0$

xxii. $\tan(2\theta) + 2\cos\theta = 0$

xxiii. $\sin\theta + \cos\theta = 1$

xxiv. $\cos(2\theta) + \sin^2\theta = 0$

xxv. $x^2 - 2\sin(2x) = 3x$

xxvi. $6\sin x - e^x = 2, x > 0$