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Domain & Range of Trigonometric Functions & Inverses

There are six trigonometric functions, and six inverse functions:

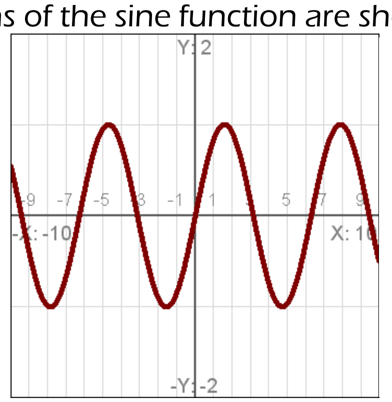
- Sine, sin(x)
- Cosine, cos(x)
- Tangent, tan(x)
- Cotangent, cot(x)
- Secant, sec(x)
- Cosecant, csc(x)

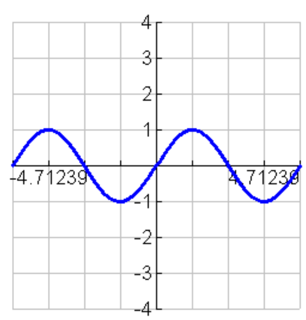
  - Inverse sine, $\sin^{-1}(x)$, arcsin(x)
  - Inverse cosine, $\cos^{-1}(x)$, arccos(x)
  - Inverse tangent, $\tan^{-1}(x)$, arctan(x)
  - Inverse cotangent, $\cot^{-1}(x)$, arccot(x)
  - Inverse secant, $\sec^{-1}(x)$, arcsec(x)
  - Inverse cosecant, $\csc^{-1}(x)$, arccsc(x)

Of these, cotangent and cosecant are the least used, and they work very much like secant and tangent, so while we will mention them, they will be glossed over somewhat.

Let's start with the six primary trigonometric functions.

1. **Sine**
   Two graphs of the sine function are shown below:

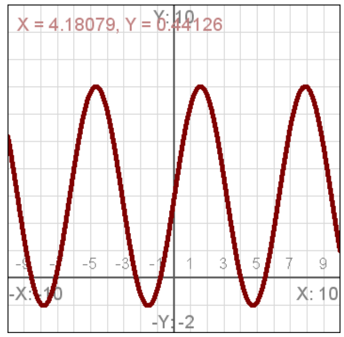




These graphs show that the domain of the sine function is all real numbers, since the oscillating pattern repeats forever. One consequence of that is that any sine function, regardless of the period or phase shift (any horizontal transformation), i.e. $\sin(\omega x+\varphi)$, will also have a domain of all real numbers.

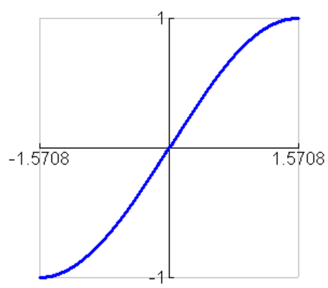
Things will not be so simple when we deal with the inverse sine function.

The range of the unaltered sine function is [-1,1] or $-1 \le y \le 1$. If we apply any vertical transformations to the graph, we must apply these to the range. The graph of Asin(x)+b transforms the range by multiplying both endpoints by A, and adding b. Therefore, the range becomes [-A+b,A+b] or $-A+b\le y\le A+b$. For example, the graph of 4sin(x)+3 has a range of [-4+3,4+3] or [-1,7]. We can see this easily from the graph of the equation shown below. Horizontal transformations have no effect on the range.



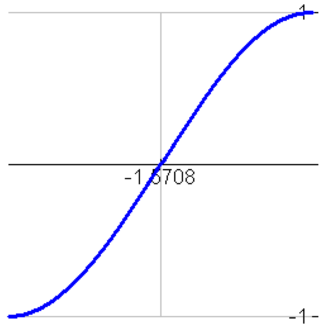
2. **Inverse Sine Function**

For inverse functions of 1-to-1 functions, it's possible to just flip the roles of the domain and range. However, as we can see from the graph of the sine function, the graph is not 1-to-1 (i.e. it fails the horizontal line test) over its entire domain. Therefore, even to define an inverse which is a function, we have to restrict the domain of the original sine function so that it is 1-to-1 over that domain. The section of the sine graph chosen is the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.



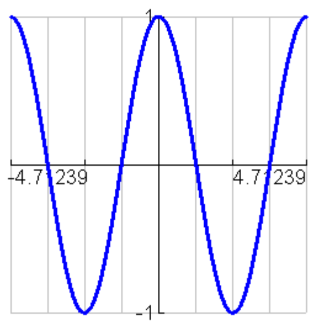
Therefore, in order to find the domain of a general sin function sin($\omega x+\varphi$), any horizontal transformations will have to be applied to this interval. Recall the horizontal transformations are "opposite" of vertical transformations. Therefore, we will divide the interval by ω, and subtract φ/ω to find the new restricted domain.

For example, the function sin(2x+π), we have a horizontal compression of a factor of 2, and a horizontal shift left of π/2. Our restricted domain, therefore is: $\left[-\frac{\pi}{4}-\frac{\pi}{2},\frac{\pi}{4}-\frac{\pi}{2}\right]=\left[-\frac{3\pi}{4},-\frac{\pi}{4}\right]$. The graph is shown below.



When we find the domain and range of the inverse function is the restricted domain that becomes the range of the inverse function, and the range that becomes the domain of the inverse function. The range of the inverse function is never all real numbers when working with trig functions.

3. **Cosine**
   The cosine function works much like the sine function does, since it's essentially the same shape graph, but with a horizontal shift of π/2. The graph below of sin(x+π/2) is identical to the graph of cos(x).



The same rules then apply to the domain and range of this graph as with sine. The domain is all real numbers, and the range of the base graph is [-1,1]. Horizontal shifts cannot alter the domain, but vertical shifts should be applied to the base range as with the sine function. Acos(x)+b has a range of [-A+b,A+b].

4. **Inverse Cosine**
   Like the sine function, the cosine function is not 1-to-1. Therefore, the domain must be restricted to a region in which is 1-to-1 (passed the horizontal line test). The interval used for the cosine for this purpose is [0,π]. To find the range of the inverse cosine, we apply the horizontal transformations from the original cosine function to this restricted domain. As before, we divide by ω, and subtract φ/ω from both endpoints of the restricted domain.

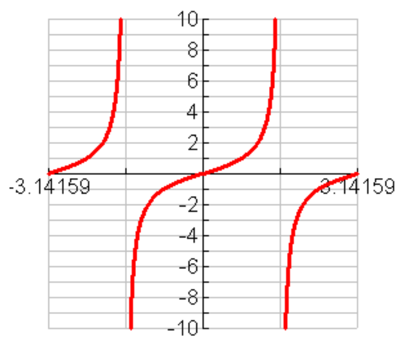
Consider the example f(x)=2cos(3x-2)+1. Find the domain and range of the inverse. We do this by finding the restricted domain and the range of the original function. The restricted domain of the original=the range of the inverse, which is: $\left[\frac{0}{3}+\frac{2}{3},\pi+\frac{2}{3}\right]=\left[\frac{2}{3},\pi+\frac{2}{3}\right]$. The range o the original=the domain of the inverse, which is [-1*2+1,1*2+1]=[-2+1,2+1]=[-1,3].

If we are working from the inverse cosine function already, then we apply horizontal transformations of the inverse (what were vertical transformations of the original cosine) directly to the domain of the inverse [-1,1] in this case, and vertical transformations (horizontal transformation of the original cosine) directly to the range of the inverse, [0,π] in the case of the cosine.

Consider: $\frac{1}{2}\arccos(x-3)+\frac{\pi}{2}$. The vertical transformations apply to the range (1/2 and π/2), and the horizontal one applies to the domain (3). The domain therefore is [2,4], and the range is [π/2,π]. We can see how this related to the original cosine function by setting this equal to y and finding the inverse (cosine). We get: cos(2x-π)+3.

5. **Tangent**

The tangent, like all of the remaining trig functions we have to talk about, is the first to have asymptotes in the graph. The graph of the tangent function is shown below.



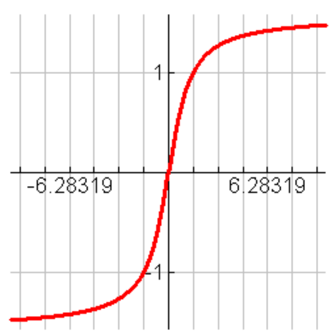
The tangent function can be defined as sin(x)/cos(x), so the asymptotes will occur where the cosine function is 0, i.e. at odd multiples of π/2. These values will have to be removed from the set of all real numbers. We can formally define the domain as: $\left(\frac{(2k-1)\pi}{2},\frac{(2k+1)\pi}{2}\right)$, for k in the set of integers.

The range of the tangent, function, unlike the sine and cosine graphs, is all real numbers.

For the domain and range of the tangent graph we have a bit of the reverse situation going on that we had in the sine and cosine graphs. Vertical transformations cannot change the range of the tangent, since the tangent is all real numbers. Horizontal transformations must be applied to the domain.

Consider the example $f(x) = 4\tan\left(2x + \frac{\pi}{4}\right) - 1$. If we want to find the domain of this function we begin with the general domain and apply the horizontal transformations, recalling that we divide by ω and here subtract π/8: $\left(\frac{(2k-1)\pi}{4} - \frac{\pi}{8}, \frac{(2k+1)\pi}{4} - \frac{\pi}{8}\right) = \left(\frac{(4k-3)\pi}{8}, \frac{(4k+1)\pi}{8}\right)$. Set k=0 to find the principle interval, $\left(-\frac{3\pi}{8}, \frac{\pi}{8}\right)$. As we said before, the range is always all real numbers and is not altered by the vertical transformations.

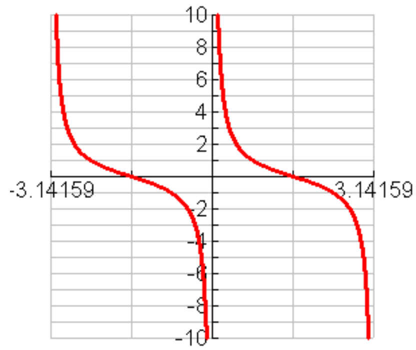
6. **Inverse Tangent**
None of the trigonometric functions are 1-to-1 and tangent is no exception. Therefore, in order to create an inverse function for tangent, the domain is restricted to one complete cycle. (Sine and cosine functions were restricted to just half a cycle.) The principle interval for the restricted domain is obtained when k=0, or for $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. (Unlike the sine function, the endpoints are not included because of the asymptotes.) The range of the tangent function on this restricted domain remains unaltered. The graph of the inverse tangent function is shown below.



In some ways, finding the domain and range of a general tangent graph is easier than with sine and cosine. Sine we already have an equation to eliminate the asymptotes, we just have to set k=0 in the tangent function to find the range of the inverse tangent function. Like sine and cosine, the range maps directly onto the domain of the inverse function. The graph's lack of continuity is what makes it appear more like what we've dealt with in past discussions of functions and their inverses.

7. **Cotangent**
The cotangent function is not normally found in calculators, so to see the shape of the function you'll need to graph it in terms of one of the available functions

(normally sine, cosine and tangent). Typically, 1/tan(x) is used. The cotangent graph cot(x) is also equivalent to $-\tan\left(x-\frac{\pi}{2}\right)$. As with the tangent function, the graph of cotangent has asymptotes, here at multiples of π, that need to be eliminated from the set of all real numbers. The principle cycle is (0,π), and so the whole domain is $(k\pi,(k+1)\pi)$, for k an integer. The principle cycle is found when k=0.



As with the tangent function, the range is all real numbers.

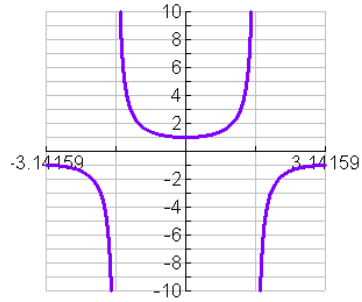
8. **Inverse Cotangent**

Like the cotangent itself, the inverse cotangent is rarely used. The base domain of the inverse cotangent is the range of the cotangent, and the range is the principle interval of the restricted domain. To obtain the graph of the inverse cotangent, it's necessary to graph the inverse of $-\tan\left(x-\frac{\pi}{2}\right)$. However, when finding values for the inverse cotangent on your calculator, we typically just use the inverse tangent function by arccot(x)=arctan(1/x). Doing this will not give us values inside the interval of (0,π), but rather, angles in the range of the inverse tangent graph, on $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

9. **Secant**

The secant function is defined as the reciprocal of the cosine function (not to be confused with the inverse function). It, therefore, like the tangent function, which can also be defined with a cosine in the denominator, has asymptotes at odd multiples of π/2. The domain, then, is the same as that of the tangent function: $\left(\frac{(2k-1)\pi}{2},\frac{(2k+1)\pi}{2}\right)$. Horizontal shifts are applied to the secant function in the same way as described for the tangent.

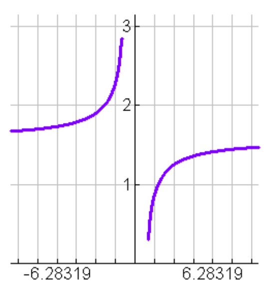
What differentiates the secant graph from the tangent graph is that the range is not all real numbers. The graph is shown below. The cosine graph is valued (in magnitude) from 0 to 1, when one takes the reciprocal of these values, the 1 stays as 1, the zero becomes the asymptote, and the values in between become larger (the reciprocal of ½ is 2, the reciprocal of 1/10 is 10, etc.) approaching the asymptote at infinity. The negative values behave similarly on the other side of the x-axis. The range, therefore, is (-∞,-1]U[1,∞).



10. Inverse Secant

The inverse secant has a domain that is the same as the range of the secant. The range, like with other trig functions, is based on a restricted domain in which one instance of each value of the graph can be obtained. This interval is the same as the restricted domain of the cosine graph: [0,2π], except that the asymptote at π/2 occurs in the middle and must be removed. The range for the inverse secant is then [0,π/2)U(π/2,π].

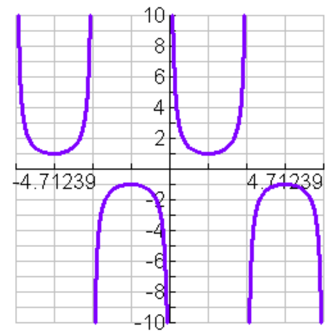
To obtain a graph of the inverse secant: arcsec(x)=arccos(1/x). The graph is shown below.



The arcsecant isn't used much in trigonometry proper, but it is used more commonly in calculus.

11. Cosecant

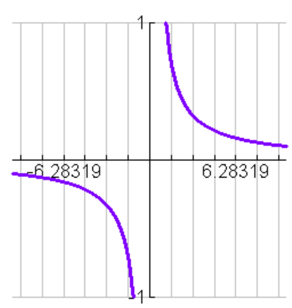
The cosecant works much like the secant, defined as 1/sin(x). Asymptotes of the graph appear at multiples of π, and therefore its domain is identical to the domain of the cotangent graph: $(k\pi,(k+1)\pi)$ for any integer k. The graph of the cosecant is shown below. Like the secant graph, the range of the cosecant is (-∞,-1]U[1,∞), for the same reasons. Horizontal transformations affect the domain, and are applied as with other graphs, dividing by ω, and shifting by φ/ω in the opposite direction of the sign. When the vertical transformations are applied to the range, the endpoints at infinity cannot be modified, but the one and negative one values will be altered.



12. Inverse Cosecant

A much neglected trig function! Like the secant, it is usually replaced in calculator land with arcsin(1/x). As shown below.

As with other trig graphs, the domain of the original function must be reduced to one complete cycle (set k=0 for the primary cycle), but with the asymptote removed, thus $[-\pi/2,0)\cup(0,\pi/2]$ in order to create a 1-to-1 function on that domain so that an inverse exists. This becomes the range of the inverse function shown below.



The domain of the inverse function is the range of the original function: $(-\infty,-1]\cup[1,\infty)$. As with graphs of other inverse functions, horizontal transformations of the original graph can be applied to the restricted domain to find the range of the inverse (dividing by ω and shifting opposite the sign of φ/ω). Vertical transformations applied to the original graph can be applied directly to the range of the original function to obtain the domain of the inverse graph.

Problems.

For each of the functions below, state the domain and range of the function. Find the inverse function, and state its domain and range.

a. $F(x)=\text{Sin}(2x)$
b. $G(x)=3\cos(x)$
c. $H(x)=4\sin(x-\pi)$
d. $J(x)=5\tan(x)$
e. $K(x)= -3\cot(2x)$
f. $L(x)=9\sec(x+1)$
g. $M(x)= -2\csc(4\pi x)+1$
h. $N(x)= -\sin(\pi x)-3$
i. $P(x)= 4\cos(-x-3\pi/2)-2$
j. $Q(x)= -\tan(x)+\pi/4$
k. $R(x)= \arcsin(1/2x+1)$
l. $S(x)= -\arccos(x-2)+\pi$
m. $T(x)= \arctan(x)+1$
n. $U(x)=\text{arccot}(3x-1)+\pi/2$
o. $V(x)= \text{arcsec}(-x-1)+\pi/4$
p. $W(x)= 1/4\,\text{arccsc}(1/2x)$