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Conic Sections

There are four conic sections: 1) the circle, 2) the parabola, 3) the ellipse, 4) the hyperbola.

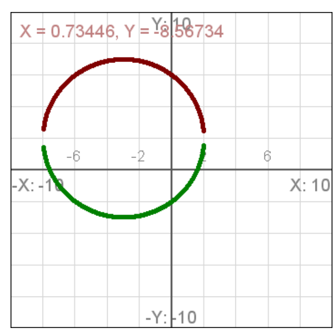
1. **The Circle** is defined geometrically to be the set of all points that are equidistant from a fixed center. We call this common distance the radius of the circle. The equation of a circle is found by using the distance formula, with our two points being (h,k) as the center, and (x,y) being any point on the circle. Then the distance (radius) is $r=\sqrt{(x-h)^2+(y-k)^2}$, and if we square both sides to get rid of the square root, we obtain the **standard form of the equation of a circle**: $(x-h)^2+(y-k)^2=r^2$. When the center is at the origin (0,0), this equation is simply $x^2+y^2=r^2$.

   Properties of the circle:

   a. The circle is not a function. We can solve for y and taking either only the positive square root or the negative one, we get one of two functions, each a semicircle; however, the entire circle itself violates the vertical line test.
   b. A circle centered at the origin is symmetric in all three directions: *x*-axis symmetric, *y*-axis symmetric and origin symmetric. Moving the circle horizontally or vertically alone destroys two of these symmetries. Any transformation that moves it both horizontally and vertically has none of the original symmetries.
   c. Circles are defined by just two numbers, the radius and the center. If you have the center and any point on the circle itself, you can find the equation of the circle. If you have two points on a diameter, you can use these to find the center. More information is needed if you have any two unrelated points on the circle.
   d. The **general form of the circle** is $x^2+y^2+Cx+Dy+E=0$ or sometimes $Ax^2+By^2+Cx+Dy+E=0$; however, in the second case, *A*=*B* or the equation is not a circle. In such a case, you can divide by *A* to obtain the prior version. To find the center and radius, one will need to complete the squares for *x* and *y* to get the equation in standard form. (When *A*≠*B*, this is an ellipse and will be dealt with in part 3.)

   Examples.

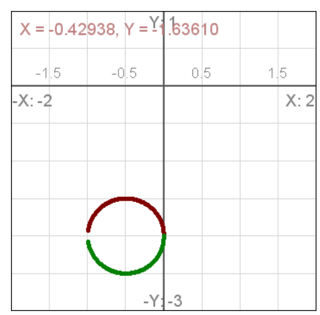
   e. *Find the equation of a circle with the radius 5, and center at (-3,2).* Here, *r*=5, *h*= -3, and *k*=2. If we put these into the standard form of the equation, we get $(x-(-3))^2+(y-2)^2=5^2$, or simplifying, $(x+3)^2+(y-2)^2=25$. The graph of the circle is shown here. Since our calculators can only draw functions, the graph is produced from the two semicircular graphs obtains from solving for *y*. The slope is vertical at the endpoints and so doesn't graph well.



   f. *Find the symmetry of the equation* $x^2+y^2=4$. We can check the symmetriy algebraically by replace *x* or *y* with their negative counterparts. For *x*-symmetry, replace *y* with (-*y*): $x^2+(-y)^2=4$, the negative will cancel out when you square it, just as $(-3)^2$ is +9, giving us back the original equation. Similarly, for *y*-axis symmetry, we replace *x* with (-*x*): $(-x)^2+y^2=4$, and as above, the negatives are cancelled by the squaring. And lastly for origin symmetry, both *x* and *y* are replaced with (-*x*) and (-*y*) respectively: $(-x)^2+(-y)^2=4$. However, any graph with both *x*- and *y*-symmetry independently is also always origin symmetric as well (this does not work in reverse).

g. *Given that the points (-1,2) and (3,6) are the endpoints of a diameter, find the equation of the circle*. First, we must find the center. This would be found from the midpoint of the diameter. $\left(\frac{-1+3}{2},\frac{2+6}{2}\right)=(1,4)$. We also need the distance between the center and either of the two points. $r=\sqrt{(1-3)^2+(4-6)^2}=\sqrt{(-2)^2+(-2)^2}=\sqrt{4+4}$ $=\sqrt{8}$. Since we are going to square after putting it in the formula, there is no need to simplify the radical further. Now that we know that $r=\sqrt{8}$, and the center $(h,k)=(1,4)$, we put these into the standard form as we did in the first example: $(x-1)^2+(y-4)^2=8$.

h. *Given the general form of the circle* $x^2+y^2-x+4y+4=0$*, find the standard form, the center, the radius and graph the equation.* To find the standard form, we must first complete the square. We first move any constants to the right to the right. $x^2+y^2-x+4y=-4$ Collecting the x's: x^2-x, we take the coefficient of the linear term, divide it by two and square it to obtain the constant we need to complete the square. Here: $\left(\frac{1}{2}\right)^2=\frac{1}{4}$. Add that to both sides: $\left(x^2-x+\frac{1}{4}\right)+y^2+4y=-4+\frac{1}{4}$ and repeat the process with the *y*'s: $\left(x^2-x+\frac{1}{4}\right)+\left(y^2+4y+4\right)=-4+\frac{1}{4}+4$. Writing each of our perfect squares in squared form, and combining on the right be obtain: $\left(x-\frac{1}{2}\right)^2+(y+2)^2=\frac{1}{4}$. Our radius is thus $\frac{1}{2}$, and the center is $\left(\frac{1}{2},-2\right)$. The graph is shown here.



Problems.

i. Find the equation of the circle in standard form with the center at (3,8) and a radius of 1.

j. Find the equation of the circle in standard for with the center at (-5,-4), with a radius of 3/2.

k. Find the equation of the circle in general form with the center at (1,-2) and a radius of 4.

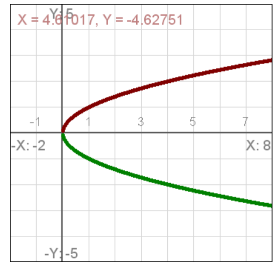
l. Find the equation of the circle in standard form with a center at (-4,-1) and a point on the circle at (9,2).

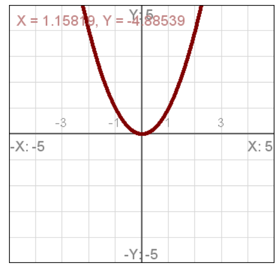
m. Find the equation of the circle in standard form with endpoints of a diameter at (-3,4) and (5,-2).

n. Find the center and radius of the circle given by the equation $x^2+y^2+4x+6y-18=0$.

o. Find the center and radius of the circle given by the equation $4x^2+4y^2+8x-6y-14=0$.

2. **The Parabola** is defined geometrically as the collection of points that are equidistant from a point (called the **focus**) and a line (called the **directrix**). From this relationship, we derive two basic forms of the parabola: $(x-h)^2 = \pm 4a(y-k)$ or $(y-k)^2 = \pm 4a(x-h)$. In the former case, the graph is a function, with the parabola opening up or down (depending on the plus or minus sign), and in the latter case the parabola opens left or right, and is not a function. The values (*h,k*) is the vertex of the graph (in both examples shown it is the origin (0,0)). The value *a* is the distance (in absolute terms) to the focus from the vertex (and in the opposite direction to the directrix). Any equation with a one variable squared and one variable linear is a parabola, thus either $Ax^2 + Cx + Dy + E = 0$ or $By^2 + Cx + Dy + E = 0$. The squared term always indicates which orientation the graph has, either up/down or left/right. Which of those can only be determined when the equation is solved for standard form.



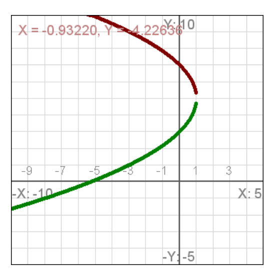


Properties of the parabola.

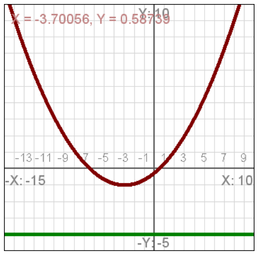
a. The focus is always inside the curve of the parabola, so that in the graph that opens up, the coordinate of the focus is (*h,k+a*), but if it opens down it is (*h,k-a*). In the graph that opens to the right, the focus is (*h+a,k*), and in the graph that opens to the left, the focus is at (*h-a,k*).
b. The directrix of the parabola is in the opposite direction of the focus. The Directrix is a line, so in the case of a parabola that opens up, it is a horizontal line *y=k-a*, or if it opens down, *y=k+a*. If the graphs opens to the right, the line is vertical, and is given by *x=h-a*, and if it opens to the left, it's *x=h+a*.
c. The axis of symmetry crosses through the vertex and the focus, and is perpendicular to the directrix.
d. The distance between the focus and the directrix is 2*a*.
e. If you have the center, and a point on the graph, you can find a by plugging in the points and solving for the missing variable.
f. All the conic sections, other than the circle can be rotated in the plane. This handout does not cover these possibilities, but you can generally recognize it by the equation containing an *xy* term.
g. Parabolas whose axis of symmetry is either the *x*- or *y*-axis can preserve its symmetry on that axis through transformation along that axis. Any other type of transformation will destroy all traditional symmetries.
h. The parabola is unique among the conics in that the "center" (*h,k*) is actually on the graph.

Examples.

i. *Find the equation of the parabola with a vertex at (2,5) and a focus at (1,5)*. The distance between the vertex and the focus is 1 in the *x* direction, and the focus is the left of the vertex, so *a*=1 and the graph opens to the left. Thus, the equation is $(y-5)^2 = -4(x-2)$. The directrix is at *x*=3.

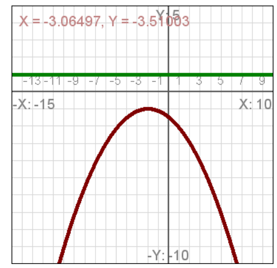


j. *Find the equation of the parabola with a vertex at (-3,2) and a directrix at y= -4*. The distance in the *y* direction is 6 between



the focus and the directrix. Thus, 6=2a, and a=3. A distance of 3 in the *y* direction from the focus (-3,2) is (-3,-1). This is vertex, (h,k). Because focus is above the vertex, the graph opens up. Putting these into the equation, we get $(x+3)^2 = 4(3)(y+1)$ or $(x+3)^2 = 12(y+1)$. The graph is shown here, along with the directrix.

k. *Find the equation of the parabola in standard form given* $x^2 - 4x + 8y + 12 = 0$. We need to start by completing the square for *x*. $(x^2 + 4x + 4) + 8y + 12 = 4$. Then move all the other terms to the right side. $(x+2)^2 = -8y - 12 + 4$, or $(x+2)^2 = -8y - 8$. Factor out the coefficient of the *y*. $(x+2)^2 = -8(y+1)$. Therefore, we know that *a*=2, since 4*a*=8. We know that the center is at (-2,-1). And we know the that graph opens down because the *x* is the squared term, and because of the negative. The focus is at (-2,-3), and the directrix is at *y*=1.



Problems. For each of the problems below, find the equation in standard form, the vertex, the focus, the directrix and use this information to graph the equation. (Hint: begin by graphing what you know.)

l. Vertex at (3,4) and focus at (3,2).
m. Vertex at (3,4) and focus at (0,4).
n. Vertex at (3,4) and focus at (3,8).
o. Vertex at (3,4) and focus at (5,4).
p. Vertex at (-1,-5) and directrix at *y*=3.
q. Vertex at (-1,-5) and directrix at *x*=5.
r. Focus at (7,-6) and directrix at *x*=2.
s. Focus at (-3,-1) and directrix at *y*= -4
t. $y^2 - 8x + 4y + 12 = 0$
u. $2x^2 + 24x - 6y - 20 = 0$

3. **The Ellipse** is defined geometrically to be the collection of points where the sum of distances to a pair of fixed points (foci) remains the same. The sum of the distances is 2*a*, and from this and the distance formula, we can generate an **equation of the ellipse in standard form** $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ or alternatively $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$ (depending on which axis is longer, the horizontal in the first case, or the vertical in the second), and the **equation of the ellipse in general form** $Ax^2 + By^2 + Cx + Dy + E = 0$, where *A*≠*B*. (You will recall that in section one, *A*=*B* was the case of a circle.) In the standard equation, the center is at (h,k).

Properties.

a. In the standard equation, a represents the length of the semi-major axis (2*a* being the length of the entire major axis), which is the distance from the center in the furthest point on the ellipse. There are two points, one on either end of the major axis, represented by ($h \pm a,k$) if the major axis is horizontal, or ($h,k \pm a$) if the major axis is vertical (corresponding to the equations listed above respectively).

b. The letter b in the standard equation represents the length of the semi-minor axis (2*b* being the length of the entire minor axis), which is the distance from the center to the closest point on the ellipse. The endpoints of the minor axis are represented by ($h,k\pm b$) if the minor axis is vertical, and ($h\pm b,k$) is the minor axis is horizontal (corresponding to the standard equations listed above respectively.)

c. The major and minor axes are always perpendicular to each other.

d. The foci of the ellipse are at a distance from the center c on the major axis, where c is related to a and b through a pseudo-Pythagorean relation: $b^2+c^2=a^2$, because in the ellipse, a is always the longest of the three distances. Therefore, in a horizontally-oriented ellipse, the foci are at ($h\pm c,k$), and in a vertically-oriented ellipse, they are at ($h,k\pm c$).

e. No ellipse is a function, and to graph it, it will have to be solved for y and use the calculator to draw the two semi-ellipses.

f. Ellipses can be rotated so that they are orientated at an angle to either axis. We will not be covering that in this handout, but you can spot it by the presence of an *xy* term in the general form of the equation. Do not attempt to solve these for *y* using standard algebraic techniques (it won't work).

g. Rotation can preserve origin symmetry for ellipses if the center remains at the origin. Symmetry will not be preserved, as with the circle, if any transformations are applied. The original graph, with center at the origin, has all three symmetries. Movement along one of the axes can preserve a single plane of traditional symmetry. The ellipse is always symmetric with respect to its own major axis and minor axis.

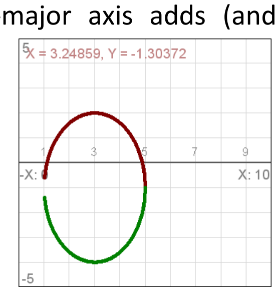
h. In the general form of the equation, we said that $A\neq B$, but for an ellipse, they must both be of the same sign (usually positive). The case where they are of opposite sign will be dealt with below in the case of a hyperbola.

i. The endpoints of the axes are sometimes referred to as vertices, but I will not do so here.

Examples.

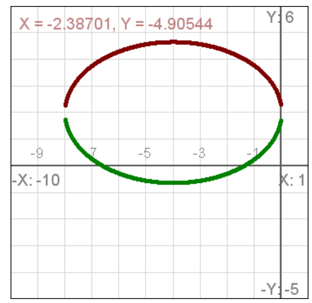
j. *Write the equation of the ellipse in standard form if the center is at (3,1) and the semi-major axis has a length of 3, the minor axis has a length of 2, and the ellipse is oriented vertically*. To solve this problem, we first need to choose the correct standard form. A graph that is oriented vertically is longer in the y-direction, and so we need the equation where *y* is divided by *a*: $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$. Since (*h,k*) is (3,1), and a (the length of the semi-major axis) is 3, and b (the length of the semi-minor axis) is 2, we plug these into the equation to get: $\frac{(x-3)^2}{4}+\frac{(y-1)^2}{9}=1$.

k. *For the problem above, find the endpoints of each axis, and the foci, and graph the equation*. Since the ellipse is oriented vertically, the semi-major axis adds (and subtracts) a from the y-coordinate of the center, giving us (3,4) and (3,-2). Similarly, the minor axis is perpendicular to the major axis, and so we add (and subtract) *b* from the *x*-coordinate of the center, giving us (1,1) and (5,1). To find the foci, we must first find *c*. Using the relation $b^2+c^2=a^2$, we have: $2^2+c^2=3^2$, or $9-4=c^2$, $c=\sqrt{5}$. Since *c* is a distance, it is strictly positive. And since the foci are along the



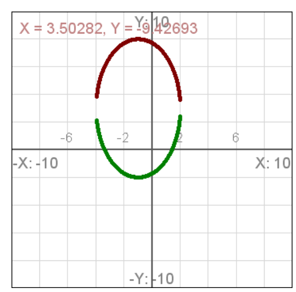
major axis, we add them to the y-coordinate of the center to get $\left(3,1\pm\sqrt{5}\right)\approx(3,3.24),(3,-1.24)$. [Use decimals only to find the approximate location of points on your graph. Always use the exact values (with the square roots) to label your values or report them as answers. Also, be sure to use the same scale in each direction of your graph, or the ellipse may not appear to have the correct orientation to do distortions in the graph.]

l. *Given a center of an ellipse is at (-4,2), and one end of the major axis is at (0,2), and one focus is at (-1,2), find the equation of the ellipse in standard form. Sketch the graph.* If the center is at (-4,2) and one end of the major axis is at (0,2), we know that the distance between them is 4, and so a=4. Similarly for the focus, giving us c=3. We can use this to find b. $b^2+3^2=4^2$ or $b^2=16-9$, and thus $b=\sqrt{7}$. Since a is the direction of the major axis and it is changing in the x-direction, we, know the ellipse is oriented horizontally. That means the equation is: $\frac{(x+4)^2}{16}+\frac{(y-2)^2}{7}=1$. To sketch the graph by hand, we will need the missing focus, the missing endpoint of the major axis, and the missing endpoints of the minor axis. These are respectively: (-7,2), (-8,2), and $\left(-4,2\pm\sqrt{7}\right)$.



m. *Given the general form of the ellipse* $25x^2+9y^2+50x-54y-119=0$*, find the standard form of the ellipse and draw the graph*. To do so, we will need to complete the squares. Collect the x-terms and y-terms together and factor out the coefficient of the squared term: $25(x^2+2x)+9(y^2-6y)=119$. To complete the x-square, we need to add 1 (times 25), and to complete the y-square, we need to add 9 (times 9): $25(x^2+2x+1)+9(y^2-6y+9)=119+25+81$.

Rewriting: $25(x+1)^2+9(y-3)^2=225$. We need 1 on the right side, so divide by whatever is there, and collect any remaining coefficients in the denominators. $\frac{(x+1)^2}{9}+\frac{(y-3)^2}{25}=1$. The center is at (-1,3), a=5 and the ellipse is oriented vertically. b=3, and we can calculate c=4. Thus, our foci are at (-1,7) and (-1,-1); the endpoints of the major axis are (-1,8) and (-1,-2), and the endpoints of the minor axis are (-4,3) and (2,3).



Problems. For each of the problems below, sketch the graph of the ellipse. For each one find the center, the foci and the endpoints of both axes.

n. $\frac{(x-2)^2}{9}+\frac{(y+1)^2}{25}=1$.

o. $\frac{(x-2)^2}{25}+\frac{(y+1)^2}{16}=1$

p. Center at (3,1), one focus at (3,4), and the one endpoint of the minor axis at (2,1).

q. Center at (-4,5), one focus at (-4,4), and the one endpoint of the major axis at (-4,1).

r. Center at (-3,-2), one endpoint of the major axis at (-3,4), and the one endpoint of the minor axis at (0,-2).
s. One focus at (0,5), and the endpoints of the major axis at (0,6) and (0,-2).
t. $3x^2 + y^2 - 2y - 2 = 0$
u. $21x^2 + 8y^2 + 84x - 84y + 44 = 0$
v. $225x^2 + 144y^2 + 450x - 576y + 701 = 0$
w. Find the domain of the graph is (o).
x. Determine the symmetry of the graph in (s).

4. **The Hyperbola** is defined geometrically as the collection of points whose distance from two fixed points (**foci**) have a fixed difference (2*a*). (This is different from the ellipse in that the ellipse was a fixed sum.) The hyperbola is actually two curves, depending on the order of subtraction. This relationship generates two possible equations of the hyperbola, depending on its orientation. If it opens up and down, we call this "transverse-*y*" because the axis that cuts through the foci and center is parallel to the *y*-axis, and the equation is: $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$. If it opens left and right, we call this "transverse-*x*" because the axis that cuts through the foci and center is parallel to the *x*-axis, and the equation is: $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$. (When the center is at the origin, it is actually these axes.). These forms are considered the **standard form**. As with the other equations, the center is at (*h,k*). The **general form** looks like that of the ellipse: $Ax^2 + By^2 + Cx + Dy + E = 0$, but A and B must be of opposite sign.

Properties.

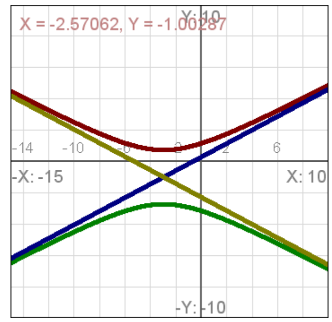
a. The negative sign determines the orientation of the graph, or more specifically, the positive term uses the variable of the transverse axis.
b. Like in an ellipse, *c* is the distance from the center to the foci, and *a* is the distance from the center to the vertex (the nearest point on the hyperbola).
c. Unlike in an ellipse, the constant a need not be bigger than *b*.
d. Of the three constants, *c* is the largest in a hyperbola, and the relationship between them mimics the Pythagorean Theorem: $a^2 + b^2 = c^2$.
e. In the general form of the equation, you can't determine the orientation of the graph until you complete the square. If after completing the square the constant on the right is negative, you will divide by that, and switch the signs.
f. The hyperbola is best drawn by drawing the two oblique asymptotic lines that it approaches. One way of drawing these lines is by drawing an imaginary rectangle defined in the transverse direction by *a*, and in the perpendicular direction by *b*. The asymptotic lines pass through the center, and the corners of these boxes. The slopes then are the vertical change over the horizontal change, and the positive and negative versions of these ratios. Either $\pm\frac{a}{b}$ or $\pm\frac{b}{a}$ depending on the orientation of the graph (transverse-y and transverse-x respectively). Thus, the equations of these lines are $y-k = \pm\frac{a}{b}(x-h)$ for transverse-*y* graphs, and $y-k = \pm\frac{b}{a}(x-h)$ for transverse-*x* graphs.

g. Hyperbolas can also be rotated, and again, the telltale sign is the cross-product term *xy*. These graphs are beyond the scope of this handout.

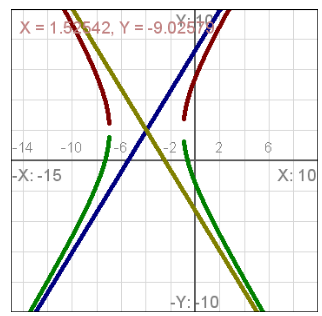
h. No complete hyperbolas are functions, but hyperbolic curves, like the top or bottom half alone of transverse-*y* hyperbolas are functions.

Examples.

i. *Graph the hyperbola given by the equation* $\frac{(y+1)^2}{3}-\frac{(x+3)^2}{16}=1$. *Find all the keypoints of the graph and the asymptotes*. First of all, we know that the center (*h*,*k*) is at (-3,-1). We also know that $a=\sqrt{3}$, and *b*=4, because remember *a* is under the positive term and need not be larger than *b*, as it was with the ellipse. Using our formula, we can calculate $c=\sqrt{19}$. Our foci then are at $(-3,-1\pm\sqrt{19})\approx(-3,3.36),(-3,-5.36)$, our vertices are at $(-3,-1\pm\sqrt{3})\approx(-3,-2.73),(-3,0.73)$, and the asymptotes are $y+1=\pm\frac{\sqrt{3}}{4}(x+3)$. The graph here shows the hyperbola, and the two asymptotes.

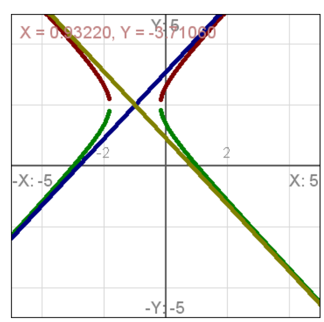


j. *Find the equation of the hyperbola with a center at (-4,2), a focus at (1,2) and a vertex at (-1,2). Graph the hyperbola, together with its asymptotes*. If the focus is at (1,2) and the center is at (-4,2), the distance between them is *c*=5. We can also find *a*, the distance to the vertex to be 3. Using our formula, we find that *b*=4. Because the focus and the vertex are changing in the horizontal direction, the graph is transverse-*x*. The equation, therefore, is: $\frac{(x+4)^2}{9}-\frac{(y-2)^2}{16}=1$. The missing vertex is (-7,2), and the missing focus is at (-9,2). The asymptotes are $y-2=\pm\frac{4}{3}(x+4)$.



k. *Find the equation of the hyperbola in standard form, and graph the hyperbola, given* $-12x^2+10y^2+24x-40y+36=0$. First, factor out the coefficients of the squared terms and collect the like variable together. $-12(x^2-2x)+10(y^2-4y)=-36$ Complete the squares by adding 1 (times -12) and 4 (times 10) to both sides. $-12(x^2-2x+1)+10(y^2-4y+4)=-36-12+40$

Write in squared form: $-12(x+1)^2+10(y-2)^2=-8$. Divide by the number on the right, since we need that to be 1. [Note that if you guessed before now that our equation was transverse-*y*, dividing by a negative here shows that it's actually transverse-*x*.] $\frac{3(x+1)^2}{2}-\frac{5(y-2)^2}{4}=1$. This is not yet in standard form, as all the coefficients must be in



the denominators. $\frac{(x+1)^2}{\left(\frac{2}{3}\right)} - \frac{(y-2)^2}{\left(\frac{4}{5}\right)} = 1$ is equivalent. Thus, $a = \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{6}}{3}$ and $b = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$. Our center is (-1,2). Our asymptotes are $y - 2 = \pm\frac{\sqrt{30}}{5}(x+1)$.

Problems. For each problem below, find the equation in standard form for the hyperbola, the foci, the vertices, the center and asymptotes (if not given), and graph the hyperbola.

l. $x^2 - y^2 = 1$

m. $(y-6)^2 - \frac{(x+1)^2}{4} = 1$

n. $\frac{(x+3)^2}{5} - \frac{(y-1)^2}{7} = 1$

o. Center at (0,3), a vertex at (4,3), and a focus at (-5,3).

p. Center at (1,2), a vertex at (1,4), and a focus at (1,10).

q. An asymptote at $y - 5 = \frac{9}{4}(x+3)$ and transverse-*y*. [Hint: remember that the slope of the asymptotes is a ratio of *a* and *b*.]

r. An asymptote at $y - 5 = \frac{9}{4}(x+3)$ and transverse-*x*.

s. $9x^2 - 4y^2 - 8y - 40 = 0$

t. $-81x^2 + 25y^2 - 324x + 150y - 504 = 0$

5. **Overview**. For each of the problems below, determine which type of graph it is: circle, ellipse, parabola or hyperbola.

a. $9x^2 + 4y^2 + 27x + 16y - 49 = 0$

b. $2x^2 + 2y^2 + 6x + 28y - 113 = 0$

c. $9x^2 + 4x - 8y + 12 = 0$

d. $3x^2 - 15y^2 - 9x + 150y - 73 = 0$

e. $25y^2 + 36x + 18y - 112 = 0$

f. $x^2 + y^2 + 6x - 24y - 36 = 0$

g. $36x^2 + 81y^2 + 168x + 99y - 85 = 0$

h. $-2x^2 - 2y^2 - 18x + 150y - 82 = 0$

i. $-2x^2 + 4y^2 - 76x + 324y - 608 = 0$