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Complex Numbers

This handout contains a complete review of complex numbers, leading up to writing complex numbers in polar form and applying DeMoivre's Theorem.

1. Imaginary Numbers

Before we introduce the notion of a complex number we must first introduce one of the elements of a complex number, an imaginary number.

Imaginary numbers are numbers which are given by the$\sqrt{-1}$ or some multiple of this. By a multiple of this, we can mean anything like $2\sqrt{-1}$or we might mean $\sqrt{-4}$ (remember that these two are the same, from properties of square roots: $\sqrt{-4} = \sqrt{4}\sqrt{-1} = 2\sqrt{-1}$). In both cases we have a negative under the square root sign. When we learned basic algebra, we learned that these numbers have no value in the real world. There are no real numbers that we can multiply together to get any negative number. When we have encountered these numbers in the past, we simply discarded them as meaningless. What we are going to do now with imaginary numbers, and then with complex numbers in pretend they exist and see where this gets us. It's amazing how much of math was originally done like this... even negative numbers and zero were mere fantasy until someone came up with a use for them. And as one can see if one ever takes physics, engineering or any higher level math courses, there are a lot of ways that imaginary and complex numbers turn out to be both useful and meaningful.

In order to help us deal with these imaginary numbers, we are going to first get rid of the square roots. Whenever we find a negative under the square root, we can always separate out the factor of $\sqrt{-1}$ as we did in the example above. Then we are going to replace that $\sqrt{-1}$ with the letter *i*. This letter *i* stands for the imaginary number, the $\sqrt{-1}$. So that now $2\sqrt{-1}$ can be written as 2i. (It should be noted here that in some disciplines, other characters are used. In engineering for instance, it's common to represent the imaginary number with a *j*, because *i* is used for current. However, we will use *i* throughout our discussion here.)

Because *i* is defined to be $\sqrt{-1}$, *i* also has another important property: $i^2 = -1$. We can use this property to determine other powers of *i*, as well as the value one obtains when multiplying numbers containing *i*.

Worked Examples.

Example 1. Rewrite the following number as an imaginary number containing i. Reduce the square roots as much as possible.

$$\sqrt{-18}$$

Remember, first factor out the $\sqrt{-1}$, giving us$\sqrt{18}\sqrt{-1} = \sqrt{9}\sqrt{2}\sqrt{-1} = 3\sqrt{2}i$. Note that the *i* is outside the square root.

Example 2. Add the following imaginary numbers: 2*i* and 3*i*.

Well, one adds 2i + 3i just like one adds 2x + 3x. They are like terms, so add the coefficients. $2i + 3i = (2 + 3)i = 5i$.

Example 3. What is i^3? i^4? i^{17}? i^{95}?

$i^3 = i^2 i = (-1)i = -i$. Use the definition of $i^2 = -1$ to reduce this problem.

$i^4 = (i^2)i^2 = (-1)*(-1) = 1$. Again, we used the definition of $i^2 = -1$. We'll use this property in the next example, too.

$i^{17} = i^{16}i = (i^4)^4 i = (1)^4 i = i$. We have an odd number, so separate out one of the *i*'s to make one of the factors even. See how the exponent divides by 4? We can use here the property from the previous example, that $i^4 = 1$ to reduce the problem further.

$i^{95} = i^{94}i = (i^2)^{47}i = (-1)^{47}i = (-1)i = -i$. As with the previous problem, we have an odd exponent, so we factor out one of the *i*'s and attempt to reduce what remains. (Notice that we are only reducing using even powers.) The exponent 94 isn't divisible by 4, so we will use the original definition to reduce this term to a power of i^2. (-1) to any odd power is still -1.

Notice that powers of *i* go through a cycle, *i*, -1, -*i*, 1, *i*, -1, -*i*, 1, ... (each number in this list is i, i^2, i^3, i^4, i^5, ...)

Example 4. Find the value of (3*i*)(4*i*).

Just as though we were multiplying (3x)(4x), we will multiply the coefficients and the 'variables'. So $(3i)4i = (3\cdot 4)*(i\cdot i) = 12i^2 = 12(-1) = -12$. That last step used the definition $i^2 = -1$.

Problem Solving Tips.

- If you feel like dealing with imaginary numbers is confusing because you don't understand what the $\sqrt{-1}$ *really is*, try not to think about it, and instead try to think of *i* as you would a variable, it's an unknown value, but it works pretty much the same as dealing with an *x*.
- When adding imaginary numbers you add them like as you would like terms, add the coefficients.
- When multiplying imaginary numbers, multiply them as one would with variables... multiply the coefficients and add the exponents of the variable.
- When reducing powers of *i*, if it's even, remember, you can divide the exponent by two, and then raise (-1) to that new power using the definition of *i.* If the exponent is odd, factor out an *i*, and then reduce the factor with the now even exponent.
- Make sure that your final answer has no power of *i* greater than 1.

Practice Problems.

1. Rewrite the following as imaginary numbers.

a. $\sqrt{-9}$ **b**. $\sqrt{-64}$ **c**. $\sqrt{-56}$ **d**. $\sqrt{-12}$ **e**. $\sqrt{-125}$ **f**. $\sqrt{-162}$ **g**. $\sqrt{-3}$

2. Add the following imaginary numbers.

a. $i + 3i$ **b**. $2i - 4i$ **c**. $\frac{1}{2}i + \frac{1}{4}i$ **d**. $\frac{1}{3}i - \frac{1}{5}i$

3. Multiply the following imaginary numbers.

a. $(i)4i$ **b**. $(-i)(-6i)$ **c**. $i^2 i^5$ **d**. $\frac{2}{3}i \cdot \frac{3}{4}i$

4. Simplify the following imaginary numbers.

a. $(i + 2i)(3i)$ **b**. $(-i - 7i)(i^2)$ **c**. $(4i)(i^2 - 3i)$

5. Reduce the following imaginary numbers as much as possible,

a. i^6 **b**. i^{13} **c**. i^{23} **d**. i^{56} **e**. i^{81} **f**. i^{107} **g**. i^{4025}

Answers

1. a. 3i, b. 8i, c. $2\sqrt{14}i$, d. $2\sqrt{3}i$, e. $5\sqrt{5}i$, f. $9\sqrt{2}i$, g. $\sqrt{3}i$. 2. a. 4i, b. -2i, c. $\frac{3}{4}i$, d. $\frac{2}{15}i$. 3. a. -4, b. -6, c. $-i$, d. -1/2. 4. a. -9, b. 8i, c. -4i -12. 5. a. -1, b. i, c. $-i$, d. 1, e. i, f. $-i$, g. i.

2. Complex Numbers.

Complex numbers are the sum of a real number (the kind we're used to working with) and an imaginary number. We usually write complex numbers in the form $a + bi$, where a and b are both real numbers, and the letter i indicates which of these elements is the imaginary number $\sqrt{-1}$. Let's take a moment to introduce some terms and notation we are going to use in referring to complex numbers.

z is the variable we use to refer to complex numbers, rather than x.

Complex numbers are made up of a *real part* and an *imaginary part*. Re(z) is the notation we use to refer to the real part of a complex number. Im(z) is the notation we use to refer to the imaginary part of a complex number. When a number is written in the form $z = a + bi$, Re(z) = a, and Im(z) = b.

The most common way we encounter complex numbers are through applying the quadratic formula to quadratic equations which have no real solution. [To refresh our memories, recall that a quadratic equation is of the form $ax^2 + bx + c = 0$. The quadratic formula helps us find the values of x that makes this equation true. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Remember that in our formula, the a, b, c refer back to the coefficients in our original quadratic equation.]

Consider the example $x^2 + x + 1 = 0$. This equation can't be factored to find a solution, so we can go to the quadratic formula to find a solution. Unlike the equation $x^2 + 3x + 1 = 0$, when we use the quadratic formula, we end up with a negative under the square root: $x = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$. Notice, that when we simplify this expression as much as we can, we end up with a real part, -1/2 in this case, and an imaginary part, $\frac{\sqrt{3}}{2}i$. Just as when we end up with real solutions, we have two solutions whenever the number under the square root is not negative. These two roots are known as complex conjugates of each other.

The *complex conjugate*, notated by $\bar{z}$, is found by making the imaginary part of z the opposite sign. So, for instance, if $z = a + bi$, then $\bar{z} = a - bi$. In our example above, one solution is $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and the other is $\frac{1}{2} - \frac{\sqrt{3}}{2}i$.

Complex numbers allow us to make generalizations about polynomials that would not be possible using just real numbers. The most important of these generalizations is the Fundamental Theorem of Algebra, which states that every polynomial of degree n has exactly n roots (counting real, complex and repeated roots).

One of the consequences of this theorem is that every polynomial with real coefficients will have an even number of complex roots. That even number may be zero, or it will be two, four, six, etc. These roots will come in pairs of complex conjugates.

We want to be able to perform basic operations on complex numbers as well, such as addition, subtraction, multiplication and division.

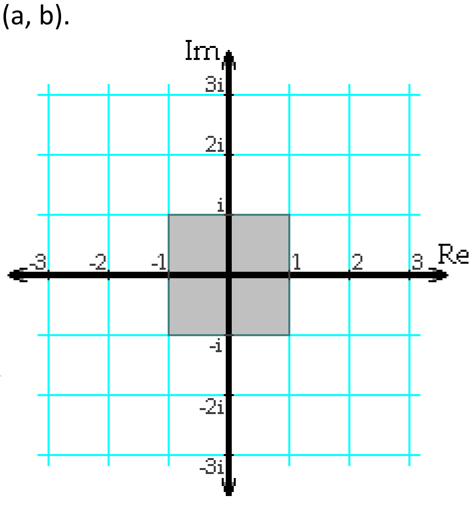
When we are doing addition and subtraction, we are going to distribute signs and add like terms just as we would as if we are using variables. For example: $z_1 = 3 - 4i$ and $z_2 = 2 + 6i$, then $z_1 + z_2 = (3 - 4i) + (2 + 6i) = (3 + 2) + (-4 + 6)i = 5 + 2i$. And $z_1 - z_2 = (3 - 4i) - (2 + 6i) = (3 - 2) + (-4 - 6)i = 1 - 10i$.

When we are doing multiplication, at first, we are also going to treat complex numbers as though we are dealing with a single variable, and FOIL. So $z_1z_2 = (3 - 4i)(2 + 6i) = 6 + 18i - 8i - 24i^2 = 6 + 10i - 24i^2$. But we aren't done, because we have to simplify i^2. Remember, $i^2 = -1$. So $6 + 10i - 24i^2 = 6 + 10i - 24(-1) = 6 + 10i + 24 = 30 + 10i$.

When we do division, we have to first remove any imaginary numbers from the denominator. Consider a number like $\frac{4}{2+3i}$. In order to remove the imaginary number from the denominator, we are going to multiply this fraction by $\frac{\overline{z_2}}{\overline{z_2}} = \frac{2-3i}{2-3i}$ where z_2 is the complex number in the denominator. One of the things we didn't talk about before when we dealt with complex conjugates is that when we multiply a complex number by its conjugate the result is a purely real number. It's just like multiplying the difference of squares formula, the middle terms drop out because they are of different signs. So for this example z_2 is 2 - 3i. So to reduce this problem, we do the following calculation: $\frac{4}{2+3i} \cdot \frac{2-3i}{2-3i} = \frac{8-12i}{4-9i^2} = \frac{8-12i}{4+9} = \frac{8}{13} - \frac{12}{13}i$.

This method of eliminating the imaginary number in the denominator is very similar to the process we use to rationalize denominators in expressions like $\frac{1}{\sqrt{2}+\sqrt{3}}$.

It is often useful to look at complex numbers visually. When we want to look at real numbers, we draw a one-dimensional number line and each point on that line represents a unique real number. But when we want to look at complex numbers, because they have both a real and an imaginary part, we are going to need not one, but two dimensions. This means that complex numbers are graphed not on a line, but on a plane. We can treat complex numbers then like we would any other point in a plane, as a Cartesian coordinate, where the x-coordinate or first coordinate is the real part or Re(z), and the y-coordinate or second coordinate is the imaginary part or Im(z). Now $a + bi$ can be represented as simply (a, b).



The complex plane then looks like the picture at left.
The corners of the center square then are, starting in the first quadrant, $1 + i$, second quadrant, $-1 + i$, third quadrant, $-1 - i$, and fourth quadrant, $1 - i$.

We will do more with plotting complex numbers in the plane when we move on to Complex Numbers in Polar and Trigonometric Form.

Worked Examples.

Example 5. State Re(*z*) and Im(*z*) for the following complex numbers.
a. z = c + d*i* **b**. z = 3 - 4*i* **c**. z = 5 **d**. z = -4*i*

a. Re(*z*) = c, Im(*z*) = d; **b**. Re(*z*) = 3, Im(*z*) = -4; **c**. Re(*z*) = 5, Im(*z*) = 0; **d**. Re(*z*) = 0, Im(*z*) = -4

We report Re(*z*) to be the part of a complex number lacking an *i*; we report Im(*z*) to be the part of the complex number which is attached to the *i*, but Im(*z*) is just the coefficient in front of the *i*, not the *i* itself. Also note the either the real or imaginary part may be 0.

Example 6. Give the real and imaginary parts of the solutions to the quadratic equation $2x^2 - x + 3 = 0$.
Use the quadratic formula to get $= \frac{-(-1)\pm\sqrt{(-1)^2-4(2)(3)}}{2(2)} = \frac{1\pm\sqrt{-23}}{4} = \frac{1}{4} \pm \frac{\sqrt{23}}{4} i$. Re(*x*) = $\frac{1}{4}$, Im(*x*) = $\pm\frac{\sqrt{23}}{4} i$.

Example 7. Give the complex conjugate of a. $z = 4 - 6i$, b. $z = 5$, c. $z = i$

To find the complex conjugate, just change the sign of the imaginary part. For a. $\bar{z} = 4 + 6i$; for b. $\bar{z} = 5$ because there is no imaginary part, it's just like 5 + 0*i*, and 5 - 0*i* is exactly the same thing; for c. $\bar{z} = -i$.

Example 8. For the equation $x^4 + 3x^3 + 6x^2 + x - 4 = 0$, how many roots does this polynomial have?
Since the highest degree term is x^4, this is a degree-4 polynomial, so it has to have 4 roots.

Example 9. Add and subtract the following complex numbers: $z_1 = 2 + 4i$, $z_2 = -7 + 3i$.

When we add we get $2 + 4i + (-7 + 3i) = (2 - 7) + (4 + 3)i = -5 + 7i$.

When we subtract, we get $2 + 4i - (-7 + 3i) = 2 + 4i + 7 - 3i = (2 + 7) + (4 - 3)i = 9 + i$.

Example 10. Multiply and divide the following complex numbers: $z_1 = 1 + i$, $z_2 = 3 - 4i$
$z_1z_2 = (1 + i)(3 - 4i) = 3 - 4i + 3i - 4i^2 = 3 - i - 4i^2 = 3 - i - 4(-1) = 3 - i + 4 = 7 - i$.

Problem Solving Tips.

- When dealing with complex numbers, there are some strategies you can use:

- If you are having trouble getting your head around what a complex number is exactly, you can think of the *i* term just like a variable, Complex numbers like 2 + 9*i* will add, subtract and multiply exactly as 2 + 9*x* does. It may help you to think of complex numbers in terms of having certain rules to follow. You'll find, after working with them for a while, that they begin to seem like they make more sense when they seem less unfamiliar. However, you can also try some links at the bottom of the page for additional explanations.

- Remember to remove complex numbers from a denominator using the complex conjugate before attempting to simplify further.

- Don't forget to replace i^2 with (-1).

❖ When you are asked for Im(z), the answer never has an i in it.

Practice Problems.

1. State Re(z) and Im(z) of the following complex numbers:

a. $4 + 6i$ **b**. $\frac{1-3i}{2}$ **c**. $-i$ **d**. 1/4

2. Find the real and imaginary parts of the solutions to the following polynomials:

a. $5x^2 - 12x - 3 = 0$ **b**. $-3x^2 + 2x - 3 = 0$ **c**. $x^3 - 1 = 0$

3. Find the complex conjugates $\bar{z}$ of:

a. $z = 1 + 0.4i$ **b**. $z = 3 - 5i$ **c**. $-.0006i$ **d**. i^2

4. How many real or complex roots do the following polynomials have?

a. $x^3 - 1 = 0$ **b**. $x^5 + 6x - 2 = 0$ **c**. $x^6 + 5x^4 - 3x^2 - x + 4 = 0$

5. Given the following complex numbers: $z_1 = 2 - i$, $z_2 = 3 + 2i$, $z_3 = 4 - 5i$, $z_4 = 2 + 4i$, $z_5 = -2 - 3i$, simplify the following expressions:

a. $z_1 + z_2$ **b**. $z_2 + z_3$ **c**. $z_3 - z_4$ **d**. $z_4 - z_5$ **e**. z_5z_1 **f**. z_1z_3 **g**. z_2/z_4 **h**. z_3/z_5

6. Plot the following complex numbers in the plane:

a. $2 - i$ **b**. i **c**. $1 + 3i$ **d**. 4 **e**. 0

Answers:
1. a. Re(z) = 4, Im(z)=6; b. Re(z) = $\frac{1}{2}$, Im(z) = $-\frac{3}{2}$; Re(z) = 0, Im(z) = -1; Re(z) = $\frac{1}{4}$, Im(z) = 0. 2. a. Re(x) = $\frac{6\pm\sqrt{51}}{5}$, Im(x) = 0; b. Re(x) = $-\frac{1}{3}$, Im(x) = $\pm\frac{2\sqrt{2}}{3}$; c. Re(x) = 1, Im(x) = 0 AND Re(x) = $-\frac{1}{2}$, Im(x) = $\pm\frac{\sqrt{3}}{2}$. 3. a. $1 - 0.4i$, b. $3 + 5i$, c. $0.0006i$, d. -1 (this number is real when it's reduced, so it's the same). 4. a. 3, b. 5, c. 6. 5. a. $5 + i$, b. $7 - 3i$, c. $2 - 9i$, d. $4 + 7i$, e. $-7 - 4i$, f. $3 - 14i$, g. $\frac{7}{10} - \frac{2}{5}i$, h. $\frac{7}{13} + \frac{22}{13}i$. 6. Plot the points with the following coordinates (2, -1), (0, 1), (1, 3), (4, 0), (0, 0).

3. Polar Form and Exponential Form

In addition to representing complex numbers in Cartesian form as a + bi, we can also represent complex numbers in polar or trigonometric form. The form gets its name because the form depends on polar coordinates containing an angle, the distance in radians from the positive x-axis, and a radius, the distance from the origin to the point in the plane. It is sometimes called trigonometric form because the form of the complex number contains a cosine and a sine. These forms of complex numbers are equivalent, but they serve different purposes. We will come to what these are later on.

If we begin with a complex number in Cartesian form, we can determine its polar form by calculating r and t, where r is the radius and t is the angle.

To find r for a complex number we, need to find the distance from the origin. We can use the distance formula, and we get the value of $= \sqrt{a^2 + b^2}$, so for the complex number z = 3 + i, $r = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$. We also sometimes see r indicated by the notation $|z|$, the modulus of z, or the length of z. These notations are equivalent. (Note that students sometimes confuse the similarity of notation with the absolute value. They are similar in that the answer is always positive, but you can't find the value of $|z|$ by just making all the signs positive.)

Then we need to find the angle *t*. To do this, we can think of the complex number forming a right triangle with the x-axis. From trigonometry, we can use the inverse tangent function to find the angle, given by $t = tan^{-1}\left(\frac{b}{a}\right)\ or\ t = \arctan\left(\frac{b}{a}\right)$.

Example 11. For the complex number 3 + 4*i*, the angle we get is $tan^{-1}\left(\frac{4}{3}\right) \approx 0.93$ radians or 53.1°. We should double check our answer to make sure our angle is in the right quadrant because it may be correct, or we may need to add π radians or 180 degrees to get it into the appropriate quadrant. (Not sure which quadrant the complex number is in? Graph it in the Cartesian plane.) This angle is sometimes referred to as the argument of *z* or arg(*z*).

To get the polar form, we now use the same transformation that we would for any parametric representation of the *x*-*y* plane. We set *x* = *r*cos(*t*) and *y* = *r*sin(*t*). But remember, the *x*-coordinate here is the real part, and the *y*-coordinate here is the imaginary part, so a + b*i* becomes *r*cos(*t*) + *r*isin(*t*). Check our answer. 5cos(.93) ≈ 2.99 and 5sin(.93) ≈ 4.01. This difference here is just a rounding error.

These forms of complex numbers also arise from another source, that is in exponential form. Exponential form and polar form are closely related through the formula: $re^{it} = r[\cos(t) + i\sin(t)]$. (This equation is referred to as the Euler equation.) This exponential form of complex numbers arises frequently in differential equations. The properties of trigonometric forms of complex numbers can be proved using the exponential form as a starting point and applying well-known properties of exponents. It is through this Euler relationship that we know that the *r* and the *t* in both forms are the same, so the same calculations can be used to determine either form. The exponential form of 3 + 4*i*, using our earlier results is $3 + 4i \approx 5e^{.93i}$.

The special properties of the polar form are derived from the properties of the exponential form. For instance, what happens when we multiply two complex numbers in exponential form together?

Example 12. Consider $(2e^{.4i})(3e^{.6i})$. We are going to multiply the coefficients and add the exponents (since the base is the same in both terms), giving us $6e^{i}$. That was a lot easier than working with these numbers in Cartesian form.

Compare: $2e^{.4i}$ = 2cos(.4) + 2isin(.4) = 1.84 + .779*i*, and $3e^{.6i}$ = 3cos(.6) + 3*i*sin(.6) = 2.48 + 1.69*i*; now multiply (1.84 + .779*i*)*(2.48 + 1.69*i*) by FOILing. 4.56 + 3.11*i* + 1.93*i* + $1.32i^2$ = 3.24 + 5.04*i*.

What's $6e^{i}$? 6cos(1) + 6isin(1) = 3.24 + 5.05*i*. The difference here is just a rounding error we could easily get rid of by carrying an extra digit.

Since exponential and polar form are the same, we can also represent this calculation in the following way. $z_1z_2 = r_1r_2[\cos(t_1+t_2) + i\sin(t_1+t_2)]$. Just as with the exponential form, we are adding the angles (the exponent portion of the exponential form) and multiplying by the length or coefficient of the exponential form. Just as we saw in the example above, the result is another calculation in polar or trigonometric form. We can carry on this generalization for any number of complex numbers we wish to multiply.

Division is similarly much easier than in Cartesian form.

Example 13. Consider $\frac{6e^{\pi i}}{2e^{\frac{\pi}{2}i}}$. When we do division of exponentials, where before we added exponents, here we subtract them, and divide the coefficients, giving us $3e^{(\pi/2)i}$. If we wanted to convert this now to Cartesian coordinates, we'd get 3cos(π/2) + 3isin(π/2) = 0 + 3i. Compare that with $-\frac{6}{2i}$.

There's no need to find the complex conjugate to solve this problem. And what did we do? Again, we can represent this process in polar form as $z_1/z_2 = (r_1/r_2)[\cos(t_1 - t_2) + i\sin(t_1 - t_2)]$. Just as in the exponential form, we subtracted the angles, and divided the coefficients. [Note: 6cos(π) + 6isin(π) = 6(-1) + *i*(0) = -6; and 2cos(π/2) + 2isin(π/2) = 0 + 2*i*(1) = 2*i*.]

One of the most powerful properties of the exponential and polar forms has to do with raising complex numbers to powers or taking roots of complex numbers. Consider a simple exponential, like e^t. If we wish to raise this to a large power, like $(e^t)^6$, we multiply the exponent by the new power, giving us e^{6t}. This is the same process we use to determine large powers of complex numbers.

Example 14. Consider the complex number 1 + *i*. If we wanted to find $(1 + i)^6$, we have two ways to approach this. We can multiply it out in some way (either by hand or by applying the binomial theorem), or we can convert it into an exponential form (or polar form as we'll see later) and raise that to the sixth power. Let's try it both ways.

Using the binomial theorem:

$$(1+i)^6 = 1^6 + 6\cdot 1^5 i + 15\cdot 1^4 i^2 + 20\cdot 1^3 i^3 + 15\cdot 1^2 i^4 + 6\cdot 1 i^5 + i^6$$
$$= 1 + 6i - 15 - 20i + 15 + 6i - 1 = -8i$$

OR $r = \sqrt{1^2 + 1^2} = \sqrt{1+1} = \sqrt{2}$ and $= tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$, thus $(1+i)^6 = \left(\sqrt{2}e^{\frac{\pi}{4}i}\right)^6$. Thus, $(\sqrt{2})^6 = 8$, and $(e^{\frac{\pi}{4}i})^6 = e^{\frac{3\pi}{2}i}$. If we convert this back through polar form, we get $8\cos\left(\frac{3\pi}{2}\right) + 8i\sin\left(\frac{3\pi}{2}\right) = 0 + 8i(-1) = -8i$.

We can do this directly in polar form through the following equation $z^n = r^n[\cos(nt) + i\sin(nt)]$. In words, raise the length of the complex number to the desired power, and multiply the angle by the desired power.

We can do roots very similarly, but where we multiply in exponential form for multiplication, we will now divide instead (since roots can be represented as fractional exponents, square root is a 1/2 exponent, cube root is a 1/3 exponent and so forth). The rooting process is most useful for determining the roots of unity, or the roots of 1.

Example 15. Consider the equation $x^5 = 1$. What are the values of *x* that make this work? Well, we know one of them straight off the bat, $1^5 = 1$, but this equation is equivalent to the expression $x^5 - 1 = 0$. We said before that because this is a fifth degree equation, there HAS to be five answers, five total values that, when we multiply the number by itself five times, we get a value of 1 at the end. So far, we've found only one of these values, what are the other four? To introduce our process, let's first start out by checking out our first value using exponential/polar form. How can we write 1 + 0*i* as a complex exponential? What is our length? Well, clearly, it's 1. What about our angle, well, it turns out to be 0, since our complex number is on the positive x-axis, and $\tan^{-1}(0) = 0$. So, $1 = 1e^{0i}$. To take the root of this number, the fifth root in this case, we are going to raise this number to the 1/5 power. $(1e^{0i})^{1/5} = 1^{1/5}*e^{0i/5} = 1e^{0i} = 1$. We get back to where we started, at 1, because 1 to any power is 1, and 0/5 is still 0.

The thing about angles, though, is that eventually you come around full circle if your angle is big enough. Big enough, in radians, is 2π. It's still true if you add another 2π to get 4π, and another 2π to get 6π and so on. In angular terms 0 = 2π = 4π = 6π = 8π. But it's exactly these angles that are going to get us our other four roots.

Redo the calculation above, but replace 0*i* with 2π*i*. $(1e^{2\pi i})^{1/5} = 1^{1/5}*e^{2\pi i/5} = 1e^{2\pi i/5} = \cos(2\pi/5) + i\sin(2\pi/5) \approx .3090 + .9511i$. (If you can run your calculator in complex number mode, you can check it out, raise this number to the fifth power and you get 1, especially if you carry several digits.)

We can repeat this process for the other angles. $(1e^{4\pi i})^{1/5} = 1^{1/5}e^{4\pi i/5} = 1\cdot e^{4\pi i/5} = \cos(4\pi/5) + i\sin(4\pi/5) \approx -.8090 + .5878i$.

$(1e^{6\pi i})^{1/5} = 1^{1/5}e^{6\pi i/5} = 1\cdot e^{6\pi i/5} = \cos(6\pi/5) + i\sin(6\pi/5) \approx -.8090 - .5878i$.

$(1e^{8\pi i})1/5 = 1^{1/5}e^{8\pi i/5} = 1\cdot e^{8\pi i/5} = \cos(8\pi/5) + i\sin(8\pi/5) \approx .3090 - .9511i$.

Those are my five roots, and notice that they come in complex conjugate pairs, just as we would have expected. We could continue like this, but if we do, we will just start to repeat the numbers we already have.

To describe this procedure directly in polar form we might write $\sqrt[n]{z} = r^{\frac{1}{n}}\left[cos\left(\frac{t+2k\pi}{n}\right) + isin\left(\frac{t+2k\pi}{n}\right)\right]$, where again we are taking just the primary root of the length of the complex number, and we are dividing the angle by number of the root. The 2kπ term refers to the process of adding multiples of 2π and repeating the process until it either starts to repeat, or we achieve the desired number of roots.

It should be noted that polar and exponential forms are less useful for adding and subtracting complex numbers. Convert to Cartesian form to perform these operations, and then convert back if need be.

Worked Examples.

Example 16. Find the polar form and exponential form of the following complex numbers:
a. 1 + *i* **b**. -*i* **c**. 3 **d**. -2 - 3*i*
For a, we find $|z| = \sqrt{2}$, and $t = tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$, thus $1 + i = \sqrt{2}\cos(\pi/4) + \sqrt{2}i\sin(\pi/4) = \sqrt{2}e^{\frac{\pi}{4}i}$.

For b, we find $|z| = 1$ and $t = \tan^{-1}(1/0)$. Tangent is undefined for the angle π/2, thus this is either that or that plus π, and since -*i* is along the negative *y*-axis, this case is the second choice, 3π/2. Thus polar form is $\cos(3\pi/2) + i\sin(3\pi/2)$ and exponential form is $e^{3\pi/2i}$.

For c, the length is clearly 3, and the angle is 0 since 3 is on the positive x-axis. The polar form is $3\cos(0) + 3i\sin(0)$, and the exponential form is $3e^{0i}$ or just 3.

For d, we find $|z| = \sqrt{13}$. And the angle is $= tan^{-1}\left(\frac{-3}{-2}\right) \approx 0.983$, but that angle is not in the third quadrant where our complex number is, so we add 3.14159 to get ≈4.12. Thus, the polar form I is ≈ $\sqrt{13}$cos(4.12) + $\sqrt{13}$isin(4.12), and the exponential form is ≈ $\sqrt{13}e^{4.12i}$.

Example 17. Multiply. **a**. $(2e^{.54i})(3.4e^{-.89i})$ **b**. 4[cos(.12) + *i*sin(.12)]·3[cos(-.7) + *i*sin(-.7)]

Multiply the coefficients and add the angles/exponents. For a, 2·3.4 = 6.8 and .54 + -.89 = -.35; thus $(2e^{.54i})(3.4e^{-.89i}) = 6.8e^{-.35i}$.

For b, 4·3 = 12 and .12 + -.7 = -.58; thus 4[cos(.12) + *i*sin(.12)]·3[cos(-.7) + isin(-.7)] = 12[cos(-.58) + *i*sin(-.58)].

Example 18. Divide. a. $\frac{4e^{\frac{3\pi}{2}i}}{2e^{\frac{\pi}{4}i}}$ b. $\frac{6\left[\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right]}{3\left[\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)\right]}$

Divide the coefficients, and subtract the angles/exponents. For a, 4/2 = 2, 3π/2 - π/4 = 5π/4; thus $\frac{4e^{\frac{3\pi}{2}i}}{2e^{\frac{\pi}{4}i}} = 2e^{\frac{5\pi}{4}i}$.

For b, 6/3 = 2 and π/6 - π/2 = -π/3; thus $\frac{6\left[\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right]}{3\left[\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)\right]} = 2\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right]$.

Example 19. Simplify. $(2 - i)^5$. State your answer in Cartesian form.

First convert 2 - *i* either to polar or exponential form. $|z| = \sqrt{2^2 + (-1)^2} = \sqrt{4+1} = \sqrt{5}$. And $t = tan^{-1}\left(\frac{-1}{2}\right) \approx -0.464$. Check the quadrant. 2 - *i*, is the point (2, -1) in the complex plane, that's in the fourth quadrant, and so is –0.464 radians, so we don't have to add any angles. The exponential form for 2 - *i* is $\approx \sqrt{5}e^{-0.464i}$ and the polar form is $\sqrt{5}[\cos(-0.464) + i\sin(-0.464)]$. In order raise these numbers to the fifth power, we raise the coefficient to the fifth power, and then multiply the angle by 5. $\left(\sqrt{5}\right)^5 = 25\sqrt{5}$, and 5(-0.464) = -2.32. We should reduce our angles as much as possible.

Generally, it's okay to use negative angles as long as they are less than π, though preferably, π/2. When they get bigger in magnitude than that, we want to, in this case, add 2π. -2.32 + 2π = 3.96. We would do the same thing if we had a positive angle larger than 2π, but then we would subtract to reduce the angle, as many times as it necessary.

Thus $(2 - i)^5 \approx 25\sqrt{5}e^{-2.32i} = 25\sqrt{5}e^{3.96i}$ and $25\sqrt{5}[\cos(-2.32) + i\sin(-2.32)] = 25\sqrt{5}[\cos(3.96) + i\sin(3.96)]$ in polar form. We can now find out what this is in Cartesian form by evaluating the polar form: 38.07 + 40.93*i*. If you have a calculator than can check the calculation for you, you get 38 + 41*i*. To avoid more of the rounding error, carry more digits.

Example 20. Simplify $\sqrt[4]{2-i}$. State your answer in Cartesian form.

First we need to covert 2 - *i* to exponential or polar form. We did this in the previous worked example. 2 - *i* ≈ $\sqrt{5}e^{-0.464i}$=$\sqrt{5}[\cos(-0.464) + i\sin(-0.464)]$. To find the fourth root, we need to take the fourth root of the coefficient, and divide the angle/exponent by four. $\sqrt[4]{\sqrt{5}} = \sqrt[8]{5} \approx 1.22$ and -0.464/4 = -0.116. Thus$\sqrt[4]{2-i} \approx \sqrt[8]{5}[\cos(-0.116) + i\sin(-0.116)]$. To get the Cartesian form, evaluate the polar form: 1.21 - .142i. If your calculator has the ability to work with complex numbers, you can check this by entering $(2 - i)^{1/4} \approx 1.21 - .141i$.

Problem Solving Tips.

- For problems involving multiplication and division, use whichever method you feel most comfortable with. If you are given Cartesian form, and like that method, then use that method. If you are familiar with the binomial theorem, you can use that method to get around converting to exponential form as well--although, remember you have to simplify and reduce all powers of i.

- You have to find two values regardless of whether or not you use polar or exponential form. Those values are the modulus of the complex number, and the angle.

- The modulus depends on the Cartesian form. It's the distance from the origin to the point in the complex plane that represents the complex number in question. You can also think of it as the hypotenuse of a right triangle with side a and b.

- You have to use the exponential or polar form to solve for roots of unity or roots in general. You can't use the Cartesian form for this.

- For roots of unity, remember that you are looking for the same number of roots as the power of the equation. There are five fifth roots, and six sixth roots, and so on. To find each root, add 2π and repeat the calculation.

- When calculating polar or exponential form, make sure that you check to see that your angle is in the correct quadrant. You may need to add π.

- When using the polar or exponential form for multiplication, you may need to add/subtract multiples of 2π to keep the angle generally between -π and 2π.

- When doing roots of unity, check that your complex answers come in pairs of complex conjugates.

Practice Problems.

1. Find the polar form of the following complex numbers; state the modulus and the argument:
a. $1 - i$ **b**. -2 **c**. $-3 + i$ **d**. $4 + 6i$

2. Multiply the following complex numbers as directed, report your answers in polar form:
a. $(1 + i)(2 - i)$ **b**. $(3e^{-4i})(6e^{3i})$ **c**. $2[\cos(.4) + i\sin(.4)]\cdot 0.1[\cos(2.3) + i\sin(2.3)]$ **d**. $(3 - i)(2e^{.01i})$

3. Divide the following complex numbers as directed, report your answers in exponential form:
a. $\frac{4-1}{2+3i}$ **b**. $\frac{7e^{\pi i}}{14e^{\frac{7\pi}{4}i}}$ **c**. $\frac{8\left[\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right]}{10[\cos(3)+i\sin(3)]}$ **d**. $\frac{1-5i}{2e^{7.1i}}$

4. Simplify. State your answer in Cartesian form.
a. $(3 + 3i)^7$ **b**. $(1 - 2i)^4$ **c**. $(5 + 3i)^8$

5. Simplify.
a. $\sqrt[3]{4-i}$ **b**. $\sqrt{3+4i}$ **c**. $\sqrt[6]{8-15i}$

6. Find the following roots of unity.

a. $\sqrt[4]{1}$, b. $\sqrt[6]{1}$, c. $\sqrt[7]{1}$, d. $\sqrt[9]{1}$.

Important Note!

While doing these examples, I have converted values to decimal places in many cases. In many cases, this is because the exact values are ugly and difficult to find, or in other cases, we've gone through the exact values and then given decimal values to compare answers. When doing problems in class, you should take care to follow directions on problems of this type and be prepared to give exact answers unless the question specifically asks you to round to a certain number of decimal places. Exact answers together with decimals is okay. Decimals instead of exact answer is not.

Answers:

1. a. $|z| = \sqrt{2}$, arg(z) = -π/4, $\sqrt{2}$ [cos(π/4) +$i$sin(π/4)]. b. $|z|$ = 2, arg(z) = π, 2[cos(π)+isin(π)]. c. $|z| = \sqrt{5}$, arg(z) = -.32175, $\sqrt{5}$ [cos(-.32175) + $i$sin(-.32175)]. d. $|z| = 2\sqrt{13}$, arg(z) = .98279, $2\sqrt{13}$[cos(.98279) + $i$sin(.98279)]. 2. a. $\sqrt{5}$ [cos(-.32175) + $i$sin(-.32175)]. b. 18[cos(-1) + $i$sin(-1)]. c. 0.2[cos(2.7) + $i$sin(2.7)]. d. $2\sqrt{5}$ [cos(-.31175) + $i$sin(-.31175)]. 3. a. $\sqrt{\frac{17}{13}}e^{-1.2278i}$. b. $\frac{1}{2}e^{-\frac{3\pi}{4}i}$. c. $\frac{4}{5}e^{-2.4764i}$. d. $\frac{\sqrt{26}}{2}e^{-2.19021i} = \frac{\sqrt{26}}{2}e^{4.0929i}$. 4. a. 17496 - 17496$i$. b. -7 + 24$i$. c. -506864 - 1236480$i$. 5. a. 1.598 - .13080$i$. b. 2 + i. c. 1.5776 - .28730i. 6. a. 1, -1, i, -i. b. 1, 1/2 + .866i, -1/2 + .866i, -1, -1/2 - .866i, 1/2 - .866i. c. 1, .62349 + .78183i, -.2225 + .97497i, -.90097 + .43388i, -.90097 - .43388i, -.2225 - .97497i, .62349 - .78183i. d. 1, .766 + .64279i, .17365 + .9848i, -1/2 + .866i, -.9397 + .342i, -.9397 - .342i, -1/2 - .866i, .17365 - .9848i, .766 - .64279i.