Taylor Polynomials

A Taylor Series is defined by the power series centered at c, representing a function f with derivatives of all order (i.e. a non-polynomial) by $\sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n = f(c)+f'(c)(x-c)+....+\frac{f^{(n)}(c)}{n!}(x-c)^n+...$. However, if c=0, this is a special case called a Maclaurin series: $\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = f(0)+f'(0)x+....+\frac{f^{(n)}(0)}{n!}x^n+...$.

The Taylor series are used to approximate functions on an interval around c. The more terms you take the better the approximation over a wider interval until, at infinity, it is exactly the same as the function.

When we omit terms, it becomes important to know how large the error is. The Error on a Taylor (or Maclaurin) series is given by $R_n(x)=\frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$ where $f^{(n+1)}(z)$ is the maximum on some interval under consideration (a point possibly different than either x or c).

Our textbook has a list of elementary power series (Taylor/Maclaurin polynomials). The series for $\frac{1}{x},\frac{1}{1+x},\ln x,\arctan x$ were either derived in, or are trivial to derive from, the material we discussed in the Power Series handout, and we won't rederive them here. Instead, we will go through some of the other functions, like the trigonometric functions, the exponential functions, etc., and some more complex ones.

As with the power series discussed previously, the advantage of such function representations is that we can have an alternate definition for complex functions. They allow us to create proofs that depend on properties of polynomials, alternate means of proving derivative relationships, and our ability to integrate expression that we would otherwise not be able to work with (or only with difficulty).

Example 1.

Find the Taylor polynomial for $f(x)=e^x$ centered at x=1 for 6 terms (i.e. n=5)

n	n!	$f^{(n)}(x)$	$f^{(n)}(c)$	(x-c)^n	$\frac{f^{(n)}(c)}{n!}(x-c)^n$
0	1	e^x	e	1	e
1	1	e^x	e	$(x-1)$	$e(x-1)$
2	2	e^x	e	$(x-1)^2$	$\frac{e}{2}(x-1)^2$
3	6	e^x	e	$(x-1)^3$	$\frac{e}{6}(x-1)^3$
4	24	e^x	e	$(x-1)^4$	$\frac{e}{24}(x-1)^4$
5	120	e^x	e	$(x-1)^5$	$\frac{e}{120}(x-1)^5$

Our Taylor Polynomial then is

$$f(x) \approx P_5(x) = e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{6}(x-1)^3 + \frac{e}{24}(x-1)^4 + \frac{e}{120}(x-1)^5 + R_5$$

Example 2.
For the function $f(x) = \sin x$, find the Maclaurin polynomial for n=6, estimate the value of the function at $x = \frac{\pi}{4}$, and find the maximum value of the error for $P_6(x)$.

n	n!	$f^{(n)}(x)$	$f^{(n)}(c)$	$(x-c)^n$	$\frac{f^{(n)}(c)}{n!}(x-c)^n$
0	1	$\sin x$	0	1	0
1	1	$\cos x$	1	x	x
2	2	$-\sin x$	0	x^2	0
3	6	$-\cos x$	-1	x^3	$-\frac{1}{6}x^3$
4	24	$\sin x$	0	x^4	0
5	120	$\cos x$	1	x^5	$\frac{1}{120}x^5$
6	720	$-\sin x$	0	x^6	0
7	5040	$-\cos x$	-1	x^7	$-\frac{1}{5040}x^7$

Therefore $P_6(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + R_6$. We use n=7 to calculate the error, and we can use $\left|f^{(n+1)}(z)\right| \le 1$, giving us $R_6 = \pm\frac{1}{5040}x^7$. You'll note that for values of x close to the center point c=0, this error will be quite small, but if we choose equivalent angles much further away from the center, like $x = \frac{17\pi}{4}$, the error will get much larger. We find that

$\sin\left(\frac{\pi}{4}\right) \approx P_6\left(\frac{\pi}{4}\right) = \frac{\pi}{4} - \frac{1}{6}\left(\frac{\pi}{4}\right)^3 + \frac{1}{120}\left(\frac{\pi}{4}\right)^5 \approx .7071430458....$, and our estimate of the error is $\pm\frac{1}{5040}\left(\frac{\pi}{4}\right)^7 \approx 3.65762... \times 10^{-5}$. We can compare this to the true value which is $\frac{1}{\sqrt{2}}$, and the difference between our estimate is $3.62646... \times 10^{-5}$, which is very close to our estimate (our estimate being slightly higher).

Practice Problems.

1. Use the sine function power series we worked out in Example 2 to verify that the derivative of sine is the cosine function by taking the derivative of $P_7(x)$ for the sine function and comparing

it to $P_6(x)$ of the cosine function (find the Maclaurin polynomial for this by the method shown in Examples 1 and 2). Are they equal?

2. Use the method shown in Examples 1 and 2 to find Maclaurin polynomial for arcsin(x). Verify that the formula $\sum_{n=0}^{\infty}\frac{(2n)!\,x^{2n+1}}{(2^n n!)^2(2n+1)}$ works as a formula to derive the terms of the series that you found.
3. We showed in the Power Series handout how to get the ln(x) series from integrating the 1/x series using the method shown in that handout. Find the Taylor polynomial centered at c=1 for the natural logarithm function and show that the two methods yield the same result.
4. Find the Maclaurin polynomial for $f(x)=(1+x)^k$, for the first four terms for some general k. Then use the result to find a power series for $g(x)=(1+x)^{-4}$ and $h(x)=(1+x)^{1/2}$. Does g(x) agree with the result of the geometric power series described in the last handout? Can you derive a general formula for the terms of h(x)?

These are all verification problems if you check the examples and the table in the textbook on page 682.

We can use these elementary functions to derive formulas for other functions, either through multiplication, or long division, or simple substitution.

Example 3.

Find a power series for $f(x)=e^{-x^2}$ centered at c=0. We find first that the Maclaurin polynomial is $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$. This is easy enough to verify for yourself using the table in Example 1, and changing c=1 to c=0 (replacing e with 1 everywhere). If we replace x in our power series with $-x^2$, we have the series we need: $f(x)=e^{-x^2}=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!}$. A transformation of $f(x)=e^{-x^2}$ is the normal curve in statistics applications. If we want to find the area under this curve, we need to integrate it. But that isn't possible with our regular antiderivative rules, or any of our available techniques. Previously, we had been forced to do this numerically (hard to do on the interval (-∞,∞)). However, with a power series, this is now just a polynomial and we can integrate easily: $\int e^{-x^2}dx=\int\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!}dx=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!(2n+1)}+C$ This series converges on the interval (-∞,∞).

Example 4.

Find a power series to represent the function $f(x)=\frac{\sin x}{x}$ and use that function to show that the $\lim_{x\to 0}\frac{\sin x}{x}=1$.

In Example 2, we found a Maclaurin polynomial for sin(x), and in summation form that is $\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$.

This formula, you'll notice, gives us only the odd powers we found in Example 2. To divide this power series by x, we divide each term by x, or divide the general formula for each term by x. This gives us

$\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!}=1-\frac{1}{6}x^2+\frac{1}{120}x^4+\ldots$ (compare to Example 2). From this, it's clear to see that when x approaches zero, this function approaches 1.

Example 5.

Find a power series to represent the function $f(x)=\sin x\cos x$ of degree 8.

Written out $\sin x = x-\frac{1}{6}x^3+\frac{1}{120}x^5-\frac{1}{5040}x^7+\ldots.$ and $\cos x = 1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\frac{x^8}{40320}-\ldots.$

We essentially want to FOIL these, but any terms that result in powers higher than 8 can be discarded.

$\left(x-\frac{1}{6}x^3+\frac{1}{120}x^5-\frac{1}{5040}x^7+\ldots.\right)\left(1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\frac{x^8}{40320}-\ldots.\right)$ = $(x)\left(1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}\right)$ +

$\left(-\frac{1}{6}x^3\right)\left(1-\frac{x^2}{2}+\frac{x^4}{24}\right)+\left(\frac{1}{120}x^5\right)\left(1-\frac{x^2}{2}\right)+\left(-\frac{1}{5040}x^7\right)(1)$ =

$x-\frac{x^3}{2}+\frac{x^5}{24}-\frac{x^7}{720}-\frac{x^3}{6}+\frac{x^5}{12}-\frac{x^7}{144}+\frac{x^5}{120}-\frac{x^7}{240}-\frac{x^7}{5040}+\ldots.$ and then adding like terms we get

$\sin x\cos x = x-\frac{2x^3}{3}+\frac{2x^5}{15}-\frac{4x^7}{315}+\ldots.$. This particular function is handy because we can use it to verify that $\sin(2x)=2\sin x\cos x$.

Practice Problems.

1. Find a Maclaurin polynomial for $\sin(2x)$ by substitution into the standard sin(x) Maclaurin series. Verify, using the information in Example 5, that $\sin(2x)=2\sin x\cos x$.
2. Find Maclaurin or Talyor polynomials (if c≠0) for the following functions to the specified degree. You can use the table in your textbook.

   a. $f(x)=e^{2x}$ $\quad 1+2x+2x^2+\frac{4}{3}x^3+\frac{2}{3}x^4+\ldots$

   b. $g(x)=\ln(x^2+1), c=1$ $\quad x^2-\frac{x^4}{2}+\frac{x^6}{3}-\frac{x^8}{4}+\ldots$

   c. $w(x)=\sec x$, degree 5. $\quad 1+\frac{x^2}{2}+\frac{5x^4}{24}+\ldots$

   d. $p(x)=\frac{1}{\sqrt{4+x^2}}$,degree 4 $\quad \frac{1}{2}-\frac{x^2}{16}+\frac{3x^4}{256}+\ldots$

   e. $q(x)=\sqrt[4]{1+x^3}$, degree 4 $\quad 1+\frac{x^3}{4}-\frac{3x^6}{32}+\ldots$

   f. $a(x)=\cos\left(3x^{1/3}\right)$ $\quad 1-\frac{9x^{2/3}}{2}+\frac{81x^{4/3}}{24}-\frac{729x^2}{720}+\ldots.$

   g. $b(x)=\cos^2 x$, degree 8 $\quad 1-x^2+\frac{x^4}{3}-\frac{2x^6}{45}+\frac{x^8}{315}-\ldots$

h. $r(x) = \sinh x$ (Hint: recall that $\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right)$, degree 5 $\quad x + \frac{x^3}{6} + \frac{x^5}{120} + \ldots$

i. $s(x) = e^{x} \sin x$, degree 8 $\quad x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} - \frac{x^6}{90} - \frac{x^7}{630} + \ldots$

j. $u(x) = \frac{e^{x}}{1+x}$, degree 5 $\quad 1 + \frac{x^2}{2} + \frac{x^3}{3} - \frac{7x^4}{24} + \frac{3x^5}{10} + \ldots$