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Integration by Substitution

Integration by substitution is a technique that allows us to find antiderivatives of functions derived via the chain rule.

Recall, the chain rule is used for taking derivatives of complicated functions created by composing two functions: $f(x)$ and $g(x)$ becomes $h(x) = f(g(x))$.

For instance, if $f(x) = \cos x$ and $g(x) = x^2 + 2$, then $h(x) = \cos(x^2 + 2)$. Following the chain rule, $h'(x) = f'(g(x)) \cdot g'(x) = -\sin(x^2 + 2) \cdot 2x$.

Integration by substitution can get us from $\int -2x \sin(x^2 + 2)dx$ back to $\cos(x^2 + 2) + C$ without explicitly knowing the result in advance.

There are two common forms of functions that can be solved by substitution. They will be

1. $\int f(x) \cdot f'(x)dx$ or
2. $\int f(g(x))g'(x)dx$

*The first case is really a special case of the second, but it shows up so often, we will treat it separately.

In case #1, we have a simple function times its own derivative, for example $\int \sin x \cos x dx$. In case #2, we have a complicated composite function times the derivative of the "inside", for example $\int 3x^2 e^{x^3} dx$ or our example from above $\int -2x \sin(x^2 + 2)dx$.

Case #1.

If your problem is of the form $\int f(x) \cdot f'(x)dx$, then let $f(x) = u$ which will be our substitution variable. If we take the derivative of this statement on both sides we get $f'(x)dx = du$ using differential notation. Replace these things in the integral: $f(x)$ with u and $f'(x)dx$ with du. We get $\int f(x) \cdot f'(x)dx = \int u du$. Now just integrate according to the power rule: $\int u du = \frac{1}{2}u^2 + C$ and then replace u with your function of x again: $\frac{1}{2}u^2 + C = \frac{1}{2}[f(x)]^2 + C$. You can check by applying the chain rule to see that you do get the expression originally in the integral.

Example 1. $\int \sin x \cos x dx$ Let $u = \sin x$ and then $du = \cos x dx$.

This implies $\int \sin x \cos x dx = \int u du = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2 x + C$.

Case #2.
Let's look at the slightly more complicated case of $\int f(g(x))g'(x)dx$. Here we want $g(x) = u$ and if we take the derivative on both sides as before we get $g'(x)dx = du$. What we get then is $\int f(g(x))g'(x)dx = \int f(u)du$. But at least now we have a simpler function to integrate and no product to worry about. Integrate with the u-function, and then be sure to replace everything back to your x-function to finish.

Example 2. a. $\int 3x^2 e^{x^3} dx$
In an exponential function, the "inside" g function is the exponent of the exponential. So let $u = x^3$, taking the derivative we get $3x^2 dx = du$. So $\int 3x^2 e^{x^3} dx = \int e^{x^3} \cdot 3x^2 dx = \int e^u du$. This is easy to integrate from here. $\int e^u du = e^u + C = e^{x^3} + C$ once we put everything back.

b. $\int -2x\sin(x^2+2)dx$
Let $u = x^2 + 2$ since that is "inside", and then $2xdx = du$. Here only the 2x is the derivative of the inside of the sine function, so when we make our substitution we will leave the negative behind. You can rearrange to make this clearer:
$\int -\sin(x^2+2)\cdot 2xdx = \int -\sin u du = \cos u + C = \cos(x^2+2) + C$ which is what we started with at the top.

Tips for choosing u.
Sometimes only experience tells you want to choose, but there are some usual suspects.

1. If you have either a log function or an inverse trig function, it must be u since we cannot obtain these functions by taking derivatives (i.e. we can't integrate them).
   a. $\int \tan x \ln(\cos x)dx$
   $u = \ln(\cos x)$ even if you don't know that $-\tan x$ is the derivative of $\ln(\cos x)$. (If it's not the derivative, we will need another technique.)

   b. $\int \frac{\ln x}{x} dx$ $\qquad u = \ln x$

   c. $\int \frac{1}{(\log_4 p)\, p} dp$
   $u = \log_4 p$ You'll want to look up how to do derivative for logs other than natural logs. In this case: $du = \frac{1}{p\ln 4} dp$

   d. $\int \frac{\arccos x}{\sqrt{1-x^2}} dx$ $\qquad u = \arccos x$

2. If you have a trig function, it's the angle inside the function that is to be replaced with u.
   a. $\int 2x\cos(x^2+2)dx$ $\qquad u = x^2+2$
   b. $\int e^x \sin(e^x+1)dx$ $\qquad u = e^x+1$
3. If you have an exponential function whose exponent is other than x (or whatever variable you are using), let the exponent be u. If you let the whole exponential be u, you will get another exponential when you take the derivative. So, if you have only one, using the whole thing won't help you.
   a. $\int 3x^2 e^{x^3} dx$ $\qquad u = x^3$
   b. $\int \sec^2 x e^{\tan x} dx$ $\qquad u = \tan x$
4. If you have powers or roots, use the inside as u. Do not use the exponents, though.
   a. $\int (x^3+2x-6)^4(3x^2+2)dx$ $\qquad u = x^3+2x-6$
   b. $\int 2x\sqrt{1+x^2}dx$ $\qquad u = 1+x^2$
5. If you have any denominators, if the numerator is one degree less than the denominator, there is a good chance (at least in the beginning here) to let the denominator be u. If there are exponents, though, see above. This can be tricky with trig functions.
   a. $\int \frac{2z}{1-z^2}dz$ $\qquad u = 1-z^2$
   b. $\int \frac{2x-2}{x^2-2x+1}dx$ $\qquad u = x^2-2x+1$
   c. $\int \frac{t^3}{t^4+1}dt$ $\qquad u = t^4+1$
   d. $\int \frac{\csc^2 x}{\cot^3 x}dx$ $\qquad u = \cot x$
   e. $\int \frac{e^x}{e^x+3}dx$ $\qquad u = e^x+3$
6. If you have a product of three or more terms, you can do a double substitution or recognize that 2 of the products are actually just coming from a double chain rule and

do it in one step. To do that latter, look for the *most complicated product*, and take the largest inside function you can.

a. $\int \tan^2\left(x^{2/3}\right)\sec^2\left(x^{2/3}\right)x^{-1/3}dx$

i. You can choose $u = x^{2/3}$ and since $du = \frac{2}{3}x^{-1/3}$ this will reduce the problem to $\frac{3}{2}\int \tan^2 u \sec^2 u du$, but then you will have to do a second substitution in order to finish integrating. Choose a different variable, let $w = \tan u, dw = \sec^2 u du$ and this will get us to a point where we can integrate finally: $\frac{3}{2}\int w^2 dw$. (we will explain the constant below)

ii. If, however, you choose $u = \tan\left(x^{2/3}\right)$ to start with, you can get to this same integration point in one step. When you take the derivative, you will get both missing pieces from the chain rule since $du = \sec^2\left(x^{2/3}\right)\cdot\frac{2}{3}x^{-1/3}dx$.

7. No substitutions can ever have your derivative (du) end up in the denominator. If it is, then your substitution is incorrect, or the integral can't be done by this method.

What if your substitution and your integral don't exactly match up?
This happens most of the time, but as long as we are only off by a coefficient (a constant multiplier) we can correct for this by tweaking the constants. It's the variable part that must match up in all cases. As long as that is true, we can adjust the others.

Example 3. $\int e^{3x}dx$

Here, we let u=3x and du=3dx. But we don't have a 3 you say? There are two ways to deal with this:

Method #1: Divide it to the other side: $\frac{1}{3}du = dx$. Now we can substitute. $\int e^{3x}dx = \int e^u \cdot \frac{1}{3}du$.
We can rearrange to move the constant out of the way, and integrate.
$\frac{1}{3}\int e^u du = \frac{1}{3}e^u + C = \frac{1}{3}e^{3x} + C$. And of course, you can always check that when you take the derivative, the factor 3 from the chain rule will cancel out and we'll get the function we started with.

Method #2: The other alternative is to multiply the entire integral by one in the form here of $\frac{1}{3}\cdot 3$, then pull the 3 inside the integral to make the substitution:

$\int e^{3x}dx = \frac{1}{3}\cdot 3\int e^{3x}dx = \frac{1}{3}\int e^{3x}\cdot 3dx = \frac{1}{3}\int e^{u}du = \frac{1}{3}e^{u} + C = \frac{1}{3}e^{3x} + C$. Either method will get us the same answer; however, I tend to prefer method #1.

Remember, though, this can only be used to adjust constant multipliers. It cannot be used to adjust variables!

Example 4. $\int \frac{x-1}{x^2-2x+3}dx$

Let $u = x^2 - 2x + 3, du = (2x-2)dx$. But that's not quite what we have in the numerator, But if we factor out a 2 and divide: $(2x-2)dx = 2(x-1)dx = du \Rightarrow (x-1)dx = \frac{1}{2}du$. Substituting now we get: $\int \frac{x-1}{x^2-2x+3}dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2-2x+3| + C$.

Example 5. $\int \frac{2}{e^x-3}dx$

Sometimes, you can do algebra to get a problem that doesn't look like it can be done with substitution into a form where it can be done. Exponentials are most susceptible to this. Here, multiply by one in the form of $\frac{e^{-x}}{e^{-x}}$. We use this form because exponentials need a second copy for the derivative. If we had an e^{-x} in the denominator, we'd multiply by $\frac{e^x}{e^x}$ instead. But in our case: $\int \frac{2}{e^x-3}dx = \int \frac{2}{e^x-3}\cdot\frac{e^{-x}}{e^{-x}}dx = \int \frac{2e^{-x}}{1-3e^{-x}}dx$ once we distribute through the denominator. Now, we can do substitution with u equal to the denominator. $u = 1-3e^{-x}, du = 3e^{-x}dx \Rightarrow \frac{1}{3}du = e^{-x}dx$ after the two negatives cancel each other out. The 2 in the numerator can just be pulled out of the integral.

$$\int \frac{2e^{-x}}{1-3e^{-x}}dx = 2\int \frac{e^{-x}dx}{1-3e^{-x}} = \frac{2}{3}\int \frac{du}{u} = \frac{2}{3}\ln|u| + C = \frac{2}{3}\ln|1-3e^{-x}| + C$$

Example 6. $\int \frac{x-4}{x^2-6x+9}dx$

Let $u = x^2 - 6x + 9, du = (2x-6)dx$. The common factor on the derivative is 2 and dividing this out we get $\frac{1}{2}du = (x-3)dx$, but this isn't exactly what we have. If we want to continue doing the problem here, we have to split the problem into two parts where the numerator adds to x-4 and one piece is x-3, i.e. $\int \frac{x-4}{x^2-6x+9}dx = \int \frac{x-3}{x^2-6x+9}dx + \int \frac{-1}{x^2-6x+9}dx$. We can do the first of these integrals by substitution, but the second piece will have to be done by completing the square and either doing a power rule or an inverse tangent integral or partial fractions, depending on whether the denominator is a perfect square (as it is here) or whether it has an extra constant (what would happen if our denominator was $x^2-6x+10$ instead).

To finish off the example here:

$$\int \frac{x-4}{x^2-6x+9}dx = \int \frac{x-3}{x^2-6x+9}dx + \int \frac{-1}{x^2-6x+9}dx =$$

$$\int \frac{\frac{1}{2}du}{u} + \int \frac{-1}{(x-3)^2}dx = \frac{1}{2}\int \frac{du}{u} - \int \frac{1}{w^2}dw = \frac{1}{2}\int \frac{du}{u} - \int w^{-2}dw = \frac{1}{2}\ln|u| + w^{-1} + C =$$

$$\frac{1}{2}\ln|x^2-6x+9| + \frac{1}{x-3} + C = \ln|x-3| + \frac{1}{x-3} + C$$

Where I let $w = x-3, dw = dx$ in the second integral, and in the last step, I just simplified using log rules.

Example 7. $\int \sec^4 x \tan x dx$

Some trig functions problems will require us to apply trig identities or pull apart powers of functions in order to do the integral. This particular example can be done in two different ways. They will look different, but they will be algebraically equivalent (plus or minus a constant).

Method #1. We may notice that $u = \tan x, du = \sec^2 x dx$. But what do we do with the other $\sec^2 x$? We replace it with $\sec^2 x = 1+\tan^2 x$, which will give us:

$$\int \sec^4 x \tan x dx = \int \sec^2 x \cdot \sec^2 x \tan x dx = \int \sec^2 x(1+\tan^2 x)\tan x dx =$$

$$\int (\tan x + \tan^3 x)\sec^2 x dx = \int u + u^3 du = \frac{1}{2}u^2 + \frac{1}{4}u^4 + C = \frac{1}{2}\tan^2 x + \frac{1}{4}\tan^4 x + C$$

Method #2. Alternatively, we may pull out a secant and get the following: $\int \sec^4 x \tan x dx = \int \sec^3 x \cdot \sec x \tan x dx$. In this case we can let $u = \sec x, du = \sec x \tan x$. This gives us $\int \sec^3 x \cdot \sec x \tan x dx = \int u^3 du = \frac{1}{4}u^4 + C = \frac{1}{4}\sec^4 x + C$. Using the same Pythagorean identity as above, you can verify algebraically the equivalence of the two results.

This example can be done by both methods because of the powers of the trig functions. Some problems of this type can only be done one way because the functions can't be completely converted. (We'll get more practice with these examples later in the course.)

Practice Problems.

1. Write each composite function $h(x)$ as $f(g(x))$ or $a(b(c(x)))$ by listing $f(x), g(x)$ or $a(x), b(x), c(x)$ as appropriate. Think about this in terms of the chain rule: how many times will you have to apply it? If just once, you have the first kind of function; if twice, the second kind.
   a. $h(x) = \sin(3x)$
   b. $h(x) = 3^{(x-2)^2}$
   c. $h(x) = \tan(e^{x^2} + 1)$
   d. $h(x) = \operatorname{csch}^2(2x)$
   e. $h(x) = \sqrt{1+6x^2}$
   f. $h(x) = \ln(13x-11)$
   g. $h(x) = \left(x^3 + 3x - 1\right)^{7/3}$
   h. $h(x) = \dfrac{1}{4x-17}$
   i. $h(x) = \sqrt{1+\sqrt{x}}$

2. For each of the $g(x)$ and $b(c(x))$ in a-i above, find $g'(x)$ and $b'(c(x))c'(x)$.
3. Integrate.
   j. $\int \sin(3x)dx$
   k. $\int (x-2)\cdot 3^{(x-2)^2} dx$
   l. $\int xe^{x^2} \tan\left(e^{x^2}+1\right)\sec^2\left(e^{x^2}+1\right)dx$
   m. $\int \operatorname{csch}^2(2x)\coth^3(2x)dx$
   n. $\int x\sqrt{1+6x^2}\,dx$
   o. $\int \dfrac{\ln(13x-11)}{13x-11}dx$
   p. $\int (x^2+1)(x^3+3x-1)^{7/3} dx$
   q. $\int \dfrac{1}{4x-17}dx$
   r. $\int \dfrac{1}{\sqrt{x}\sqrt{1+\sqrt{x}}}dx$
4. Integrate the examples from the "Tips" section.
   s. $\int \tan x \ln(\cos x)dx$
   t. $\int \dfrac{\ln x}{x}dx$

u. $\int \frac{1}{(\log_4 p)\,p}\,dp$

v. $\int \frac{\arccos x}{\sqrt{1-x^2}}\,dx$

w. $\int 2x\cos(x^2+2)dx$

x. $\int e^x \sin(e^x+1)dx$

y. $\int 3x^2 e^{x^3}dx$

z. $\int \sec^2 x e^{\tan x}dx$

aa. $\int (x^3+2x-6)^4(3x^2+2)dx$

bb. $\int 2x\sqrt{1+x^2}\,dx$

cc. $\int \frac{2z}{1-z^2}\,dz$

dd. $\int \frac{2x-2}{x^2-2x+1}\,dx$

ee. $\int \frac{t^3}{t^4+1}\,dt$

ff. $\int \frac{\csc^2 x}{\cot^3 x}\,dx$

gg. $\int \frac{e^x}{e^x+3}\,dx$

hh. $\int \tan^2\left(x^{2/3}\right)\sec^2\left(x^{2/3}\right)x^{-1/3}dx$

5. Integrate.

ii. $\int \sin x\cos x e^{\sin^2 x}dx$

jj. $\int \frac{\log_5 x}{x}\,dx$

kk. $\int \tan x dx$ [Hint: write as $\tan x = \frac{\sin x}{\cos x}$.]

ll. $\int \frac{2x-5}{1+x^2}\,dx$

mm. $\int \frac{4}{5+2e^{-x}}\,dx$

nn. $\int \frac{\operatorname{arcsec}^2 x}{x\sqrt{x^2-1}}\,dx$

oo. $\int \cos x\tan(\sin x)\sec^2(\sin x)dx$

pp. $\int \frac{\operatorname{arcsec}^2 3x}{x\sqrt{9x^2-4}}\,dx$

qq. $\int \cos^2 x \sin^3 x dx$