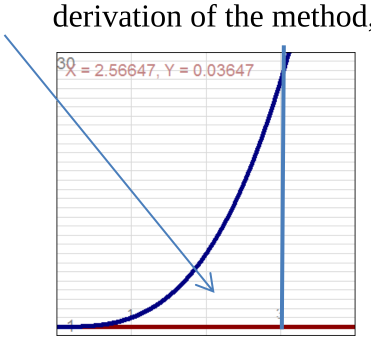
Solids of Revolution

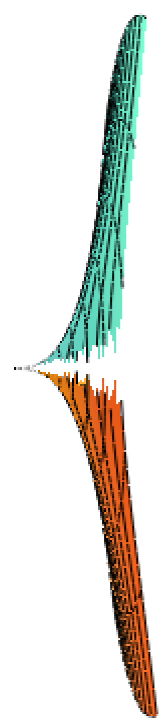
Solids of revolution are shapes formed by using a cross-sectional area and revolving the shape around an axis of rotation. Shapes of this form are quite common in manufacturing, and so it's important to be able to figure out the volume to determine how much material will be used, etc., as well as other examples such as finding the center of mass, etc. This handout is going to take two approaches. The first will involve a method with single variables designed for a course in integral calculus (Calculus II), and a second method done with multiple variables designed for a third semester or fourth quarter course in calculus (multivariable calculus). The second method will begin with the equation for the solid of revolution in three variables and proceed from there. If you are using this for a Calc II course, it's best if you skip over the advanced version of each example. Later we will discuss calculating surface areas for these same kinds of solids.

In the one-variable approach, there are two techniques available to us: The Washer Method and the Shell Method. (A third technique, the Disk Method is just a special case of the Washer Method, and so we will largely ignore it here.) In the case of the Washer Method, we will be reducing the problem to the calculation of a cylindrical slice of the shape (or a washer-shaped slice in most cases). Using the area of this slice, we will be able to integrate to get the missing dimension. In the multivariable case, we integrate in three dimensions, so we can build up the form from just a constant. The Shell Method is a bit different, as it's more closely related to polar coordinates than rectangular ones, but we will discuss the details more when we get to an example.

Solids of Revolution problems are best learned by doing examples. For a more theoretical derivation of the method, you can find this in your textbook.



Example 1. Calculate the *volume* of the solid of revolution formed by revolving the region bounded by $y = x^3, x = 3, y = 0$ around the x-axis (y=0). The region in the x-y plane looks like this. Once we revolve it, the solid is given by the equation $y^2 + z^2 = x^6$, and shown below. It looks a bit like a very squished French horn. Here, we will use the Washer Method $V = \pi\int_a^b R(x)^2 - r(x)^2 dx \text{ or } V = \pi\int_a^b R_{outer}{}^2 - R_{inner}{}^2 dx$. To do that, we need to an "outer radius" (R(x)) and an "inner radius" (r(x)). The outer radius is given by the distance to the furthest part of the region to the axis of rotation (i.e. the function minus the axis of rotation), and in this case, that's $y = x^3 - 0 = x^3 = R_{outer}$. The inner axis is given by the distance between the inner function and the axis of rotation (i.e. the function minus the axis of rotation), and in this case $0 - 0 = 0 = R_{inner}$. When $R_{inner} = 0$, the Washer Method reduces to the Disk Method where $V = \pi\int_a^b R(x)^2 dx$. This will occur



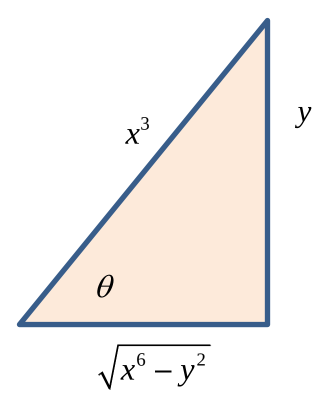
whenever the region touches the axis of rotation along its whole length, or when one side of the region is the same as the axis of rotation, as it does here. We will continue using the Washer Method, however, because the formula is more general. Note: we are integrating in x because we are rotating around the x-axis—we would do the same if we were rotating around any line parallel to the x-axis—and because we have functions of x (y is defined in terms of x).

$$V=\pi\int_a^b R(x)^2-r(x)^2\,dx=\pi\int_0^3\left[x^3\right]^2-0^2\,dx=\pi\int_0^3 x^6dx=\pi\left[\frac{x^7}{7}\right]_0^3=\frac{\pi\cdot 3^7}{7}=\frac{2187\pi}{7}$$

Example 1 Advanced. When we have multiple variables available, we can calculate this by a double or triple integral, and using the equation of the three-dimensional region: $y^2+z^2=x^6$. Solve for z: $z=\pm\sqrt{x^6-y^2}$. Thus, our double integral becomes

$V=\int_0^3\int_{-x^3}^{x^3}\sqrt{x^6-y^2}-\left(-\sqrt{x^6-y^2}\right)dydx=2\int_0^3\int_{-x^3}^{x^3}\sqrt{x^6-y^2}\,dydx$ where we are using the positive square root as the "top" function, and the negative square root as the "bottom" function. Or, as a triple integral: $\int_0^3\int_{-x^3}^{x^3}\int_{-\sqrt{x^6-y^2}}^{\sqrt{x^6-y^2}}dzdydx$. This is an even function of z, so we can reduce this integral to $2\int_0^3\int_{-x^3}^{x^3}\int_0^{\sqrt{x^6-y^2}}dzdydx$. This integral will need trig substitution if we leave it in rectangular coordinates.

Starting from $\int_0^3\int_{-x^3}^{x^3}\sqrt{x^6-y^2}\,dydx$, which is where we will end up after the first step of the triple integral anyway, x is the constant since we are integrating for y, and so let $x^3\sin\theta=y,\sin\theta=\frac{y}{x^3},dy=x^3\cos\theta d\theta$. Then $\sqrt{x^6-y^2}=\sqrt{x^6-x^6\sin^2\theta}=\sqrt{x^6(1-\sin^2\theta)}=x^3\cos\theta$. Replacing everything in the integral now we get:



$$\int\sqrt{x^6-y^2}\,dy=\int x^3\cos\theta\cdot x^3\cos\theta d\theta=x^6\int\cos^2\theta d\theta=$$
$$x^6\int\frac{1}{2}(1+\cos 2\theta)d\theta=\frac{x^6}{2}\left[\theta-\frac{1}{2}\sin 2\theta\right]$$

To get this back into terms of y, we apply the identity $\sin 2\theta=2\sin\theta\cos\theta$. This gives us $\frac{x^6}{2}[\theta-\sin\theta\cos\theta]$, and then solving for θ in our substitution, and using the triangle above, we get

$$2\int_0^3\int_{-x^3}^{x^3}\sqrt{x^6-y^2}\,dydx = 2\int_0^3 \frac{x^6}{2}\left[\arcsin\left(\frac{y}{x^3}\right)-\frac{y}{x^3}\frac{\sqrt{x^6-y^2}}{x^3}\right]_{-x^3}^{x^3} dx =$$

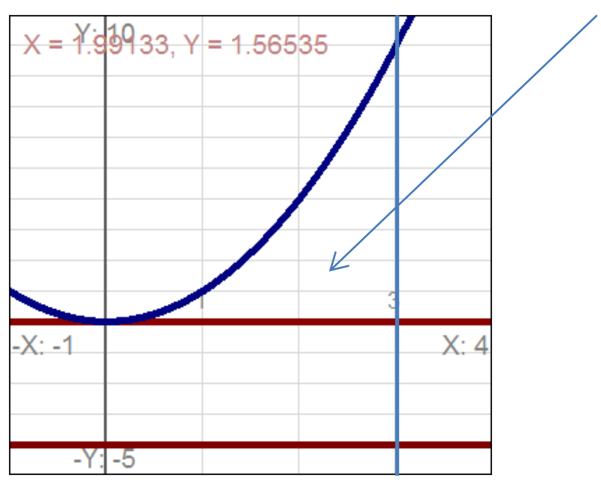
$$\int_0^3 x^6\left[\arcsin\left(\frac{x^3}{x^3}\right)-\frac{x^3}{x^3}\frac{\sqrt{x^6-x^6}}{x^3}-\arcsin\left(\frac{-x^3}{x^3}\right)+\frac{x^3}{x^3}\frac{\sqrt{x^6-x^6}}{x^3}\right]dx=\int_0^3 x^6\left[2\arcsin(1)\right]dx$$

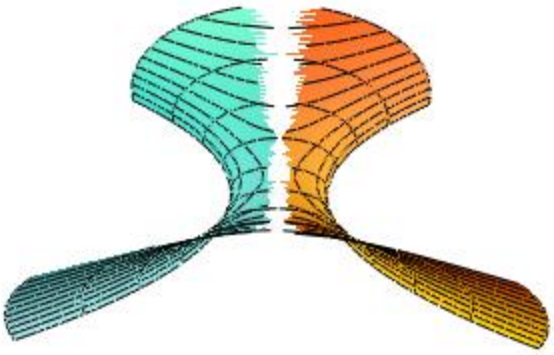
Since $\arcsin 1 = \frac{\pi}{2}$, we are left with the integral $\int_0^3 \pi x^6 dx = \frac{2187\pi}{7}$. Notice that by the last integral here, we had derived the same formula we used in the one variable case. That's how the formula for the Washer Method was derived in the first place.

We can make it a little easier if we switch variables. This volume $y^2+z^2=x^6$ has the same volume as $x^2+y^2=z^6$ by replacing x and z, with z bounded by z=0 and z=3. From here, we can convert to cylindrical or polar coordinates. $x^2+y^2=z^6 \Rightarrow r^2=z^6 \Rightarrow z=\sqrt[3]{r}$. Our double integral form them becomes $\int_0^{2\pi}\int_0^{27}\sqrt[3]{r}\cdot r\,dr\,d\theta=\int_0^{2\pi}\int_0^{27} r^{4/3}\,dr\,d\theta$, or in triple integral form $\int_0^{2\pi}\int_0^{27}\int_0^{\sqrt[3]{r}} r\,dz\,dr\,d\theta=\int_0^{2\pi}\int_0^{3}\int_0^{z^3} r\,dr\,dz\,d\theta$. Compare the result you get from integrating these to the result from the one variable example. Both all three integrals should yield $\frac{2187\pi}{7}$.

$$\int_0^{2\pi}\int_0^3\int_0^{z^3} r\,dr\,dz\,d\theta=\int_0^{2\pi}\int_0^3\left[\frac{r^2}{2}\right]_0^{z^3}dz\,d\theta=\frac{1}{2}\int_0^{2\pi}\int_0^3 z^6\,dz\,d\theta=\frac{1}{2}\int_0^{2\pi}\left[\frac{z^7}{7}\right]_0^3 d\theta=\frac{2187}{14}\int_0^{2\pi}d\theta=\frac{2187}{14}\cdot 2\pi=\frac{2187\pi}{7}$$

Example 2. Calculate the volume of the solid formed by revolving the region bounded by $y=x^2, x=3, y=0$ around the line $y=-4$. This example will be similar to example #1, but the axis of rotation is not y=0, and the washer method will not reduce to the disk method since the region and the graph do not touch. The equation of the exterior of the three-dimensional region is $y^2+z^2=(x^2+4)^2$ or equivalently $(y+4)^2+z^2=x^4$. Because the region doesn't touch the axis of rotation, there is a cylindrical hole through the center that we didn't have before, like a bore hole.





Using the Washer Method:
$R_{outer} = x^2 - (-4) = x^2 + 4$. If you think about it, this makes sense because the function is an extra 4 units away from the axis of rotation than it is from the x-axis, where we normally measure function values. Therefore $R_{inner} = 0 - (-4) = 4$. This is the gap between the area we want and what we will need to remove from the center, our bore hole. And again, we are using the Washer Method because y= -4 is parallel to the x-axis, and our function is in terms of x. (If it wasn't, we'd have to solve for y(x) if that was possible in order to use this method.)

$$V = \pi\int_a^b R_{outer}^2 - R_{inner}^2 dx = \pi\int_0^3 (x^2+4)^2 - 4^2 dx = \pi\int_0^3 x^4 + 8x^2 + 16 - 16dx = \pi\int_0^3 x^4 + 8x^2 dx =$$

$$\pi\left[\frac{1}{5}x^5 + \frac{8}{3}x^3\right]_0^3 = \pi\left[\frac{243}{5} + 72\right] = \frac{603\pi}{5}$$

Example 2 Advanced. Similarly, we can use the 3D equation for the volume to calculate the answer using double or triple integrals with the same results. As with Example 1, we switch x and z to convert to polar or cylindrical coordinates to avoid any possible trig substitution. $y^2 + z^2 = (x^2+4)^2 \Rightarrow x^2 + y^2 = (z^2+4)^2 \Rightarrow r = z^2 + 4$. Solving for z then gives: $z = \pm\sqrt{r-4}$.
Note that the interior portion we are removing is just a cylinder of radius 4 and height 3, and so to get the volume of the missing center, we can apply the geometry formula $V_{cylinder} = \pi r^2 h = \pi(4)^2 \cdot 3 = 48\pi$ and just subtract that off our total volume of the exterior shape.

Our volume integral is then: $V = \int_0^3\int_{-4}^{x^2}\int_{-\sqrt{(x^2+4)^2-y^2}}^{\sqrt{(x^2+4)^2-y^2}} dzdydx = 2\int_0^3\int_{-4}^{x^2}\sqrt{(x^2+4)^2-y^2}\,dydx - 48\pi$ in rectangular, or in cylindrical $\int_0^{2\pi}\int_0^3\int_0^{z^2+4} rdrdzd\theta - 48\pi$. Completing the integration, we get

$$\int_0^{2\pi}\int_0^3\int_0^{z^2+4} rdrdzd\theta - 48\pi = \int_0^{2\pi}\int_0^3 \frac{r^2}{2}\bigg|_0^{z^2+4} dzd\theta - 48\pi = \frac{1}{2}\int_0^{2\pi}\int_0^3 \left(z^2+4\right)^2 dzd\theta - 48\pi =$$

$$\frac{1}{2}\int_0^{2\pi}\int_0^3 z^4 + 8z^2 + 16dzd\theta - 48\pi = \frac{1}{2}\int_0^{2\pi}\left[\frac{1}{5}z^5 + \frac{8}{3}z^3 + 16z\right]_0^3 d\theta - 48\pi = \frac{1}{2}\int_0^{2\pi}\left[\frac{243}{5} + 72 + 48\right]d\theta$$

$$\frac{1}{2}\int_0^{2\pi}\frac{843}{5}d\theta - 48\pi = \frac{1}{2}\cdot\frac{843}{5}\theta\Big|_0^{2\pi} - 48\pi = \frac{843\pi}{5} - 48\pi = \frac{603\pi}{5}$$

As was expected.

Example 3. Calculate the *volume* of the solid of revolution bounded by the graph of $y = x^3, x = 3, y = 0$ (as in Example 1), but now revolved around the y-axis (x=0). The equation for the three-dimensional solid is $x^2 + z^2 = y^{2/3}$.

Alternate 1. Here, the problem is not set up ideally for the washer method. To do it by this method, we would have to match the axis of rotation (the y-axis) with the variable of integration. We'd need to solve for x as a function of y in order to do this. $y = x^3 \Rightarrow x = \sqrt[3]{y}$. Now we can use the Washer Method with dy. Our limits have also changed: when x=3, y=27. What are our inner and outer radii? The axis of rotation is to the left of the region, and so the leftmost function is now the inner radius, and the rightmost function is the outer radius. $R_{outer} = 3 - 0 = 3, R_{inner} = \sqrt[3]{y} - 0 = \sqrt[3]{y}$. It's easy to get confused here, but the region we are revolving is not touching the axis of rotation everywhere, and so it cannot reduce to the Disk Method. Do not worry too much, however, about confusing R_{outer} and R_{inner}. If you get the functions correct but subtract in the wrong order, you will get the correct magnitude and a stray minus sign, which you can neglect since volume must be positive.

Our volume integral is then: $V = \pi \int_a^b R_{outer}^2 - R_{inner}^2 dy = \pi \int_0^{27} 3^2 - \left(\sqrt[3]{y}\right)^2 dy = \pi \int_0^{27} 9 - y^{2/3} dy$. To finish it off, then: $\pi \int_0^{27} 9 - y^{2/3} dy = \pi \left[9y - \frac{3}{5} y^{5/3} \right]_0^{27=3^3} = \pi \left[243 - \frac{729}{5} \right] = \frac{486\pi}{5}$. You should ***not*** expect an area equal to what you get rotating it around the other axis.

Alternate 2. Instead of changing variables, though, which may not always be easy, we can apply the Shell Method and stay with the x variables. The nice thing about the Shell Method is that the axis of rotation and the variable of integration should be opposites. (Recall that when we are not rotating around a primary axis, it's the axis parallel to the line we are using that determines this.)

The Shell Method uses the formula $V = 2\pi \int_a^b r(x)h(x)dx$, where r(x) is the radius of the shell, normally just a variable (or a variable minus the axis-of-rotation), and h(x) is the height of the shell, which is normally just the function (or difference of functions) defining the region. In this example $r(x) = x - 0 = x$ since the axis of rotation is x=0. And $h(x) = x^3 - 0 = x^3$ since the height of the region is defined by the top function minus the bottom function (y=0). The limits are then the limits in x. This gives the volume as $V = 2\pi \int_a^b r(x)h(x)dx = 2\pi \int_0^3 x \cdot x^3 dx = 2\pi \int_0^3 x^4 dx = 2\pi \frac{1}{5} x^5 \Big|_0^3 = \frac{486\pi}{5}$ just as we had before. If you've done everything correctly, these methods should yield the same results, and it can be a great way to master both methods, and check you work when no answer keys are available. (Time-consuming, admittedly, but effective.)

Example 3 Advanced. Our three-dimensional equation for this volume is $x^2+z^2=y^{2/3}$, or if we solve for z, $z=\pm\sqrt{y^{2/3}-x^2}$. In rectangular coordinates, our triple integral is

$$\int_0^{27}\int_{-\sqrt[3]{y}}^{\sqrt[3]{y}}\int_{-\sqrt{y^{2/3}-x^2}}^{\sqrt{y^{2/3}-x^2}} dzdxdy = 2\int_0^{27}\int_{-\sqrt[3]{y}}^{\sqrt[3]{y}}\sqrt{y^{2/3}-x^2}\,dxdy$$, which we could try to do with trig substitution as we did before. However, a swap of variables as we had before will allow us to do the integral in cylindrical coordinates.

$x^2+z^2=y^{2/3} \Rightarrow x^2+y^2=z^{2/3} \Rightarrow r=\sqrt[3]{z} \Rightarrow r^3=z$. This replacement gives us a nice easy integral: $\int_0^{2\pi}\int_0^3\int_0^{r^3} rdzdrd\theta = \int_0^{2\pi}\int_0^3 r^4 drd\theta = \int_0^{2\pi}\frac{1}{5}r^5\Big|_0^3 d\theta = \int_0^{2\pi}\frac{243}{5}d\theta = \frac{486\pi}{5}$ as was expected from the single-variable method.

Let's try one last example with the Shell Method.

Example 4. Find the volume of the solid given by the region bounded by revolved around the line y=6. Our functions are in y variables and we are rotating around a line parallel to the x-axis, so we are all set up for the Shell Method integrating in y variables. We need a radius function and a height. The radius, r(y) recall is given by the variable and the axis of rotation: $r(y)=y-6$, and the height h(y) is the right function minus the left function: $h(y)=2-(y^2-4)=6-y^2$. Finally, get our limits from intersecting $x=y^2-4$ and x=2: $y=\sqrt{6}$ Putting this altogether our volume integral is

$$V=2\pi\int_0^{\sqrt{6}}(y-6)(6-y^2)dy = 2\pi\int_0^{\sqrt{6}} 6y-y^3-36+6y^2dy =$$

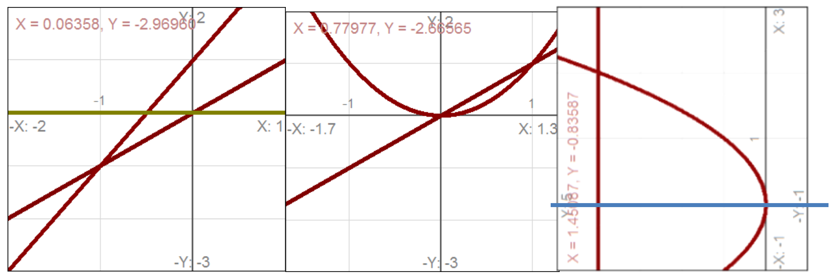
$$2\pi\left[3y^2-\frac{1}{4}y^4-36y+2y^3\right]_0^{\sqrt{6}} = 2\pi\left[18-9-36\sqrt{6}+12\sqrt{6}\right]=\pi\left[18-48\sqrt{6}\right]$$

This value is negative because the axis of rotation (y=6) is larger than any of the y values used in the integral. So, technically, we should use 6-y here to get a positive result, but we can do that at the end here by just taking the absolute value of our answer, here: $\pi\left[48\sqrt{6}-18\right]$.

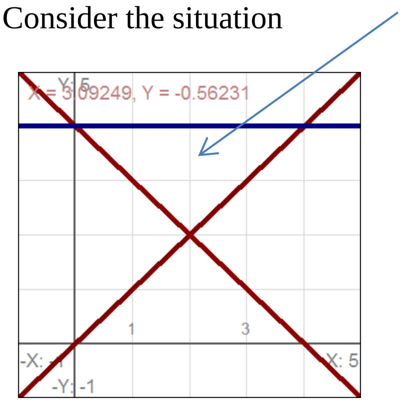
This problem can be reworked using the washer method to check the results, but I'll leave the details for the practice problems.

Practice Problems.

1. Redo Example 1 using the Shell Method and verify that you get the same result.
2. Redo Example 2 using the Shell Method and verify that you get the same result.
3. Redo Example 4 using the Washer Method and verify that you get the same result.
4. *For Advanced Students*, do Example 4 using a double or triple integral. You will probably want to switch to polar/cylindrical coordinates to avoid nasty trig substitution.
5. Use the following regions to find the volume of the solid of revolution obtained from revolving around the axes: a) the x-axis, b) the y-axis, c) the line x=5, and d) the line y=(-2).
   i. $y = x, y = 2x+1, y = 0$
   ii. $y = x, y = x^2$
   iii. $x = 4, x = y^2, y = 0$
   iv. $y = 4, x - y = 0, x + y = 4$

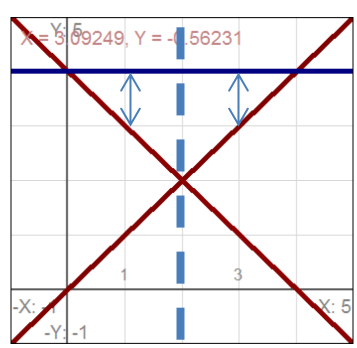


In most of these cases we've considered, we started out with equations that are in a particular variable, and except in one case, we kept them in that variable to integrate. But what if our equations aren't solved for a particular variable? How do we know which variable to use? Are there circumstances where we'd want to use one method or another? Consider the situation where our region is defined by $y = 4, x - y = 0, x + y = 4$ how do we decide which method to use?

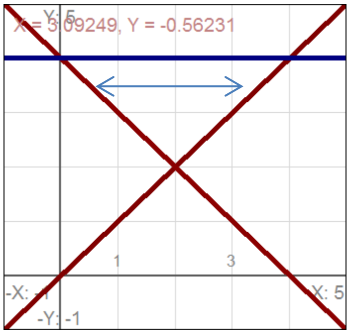


First, graph the equations. In this case, we just have nice linear equations, so that's not a problem: solve for y and put them in your calculator or plot a couple intercepts. You get the graph shown on the right. Suppose I wanted to revolve this region, say around the x-axis, which method should I use?

As it turns out, what to do here depends on figuring out in which direction, if any, the region can be done in one piece. Is there a common top function and a common bottom function for the whole region? Alternatively, is there a common right function and a common left function for the whole region? It's possible there won't be any common functions. In that case, just pick whichever variable you like and go for it piece-by-piece. But often one variable choice will let you integrate the region in one piece, but another variable will need you to do two regions. That's the situation we have here.



Look carefully at the graph again. The height in the y direction (using x variables) is a problem. The height of the region to the left of the dashed line is given by the difference between y=4 and the line y=4-x. But on the right side of the dashed line, the height of the region is determined by the difference between y=4 and y=x. That's two different sets of equations. If we want to work in x variables, then, we'd need to set up two different integrals, one for the left side up to the intersection at x=2, and one for the right side from x=2 on. This analysis does not depend on which axis we are revolving the graph around either, only which variable we are trying to use. Now, consider the same region in the horizontal direction.



In this direction, there is only one right function and one left function, so there will only be one integral. In such a circumstance, we want to work in the y variables for sure because will be less work.

So, now if we ask, which method do we use if we are revolving around the x-axis, we know that we want to work in y-variables, and so we should use the Shell Method. If we want to revolve around the y-axis, we will use the washer method, because we are still using y-variables. Even if the equations are a bit messier than this, doing one integral is going to compensate for doing a bit more algebra up front.

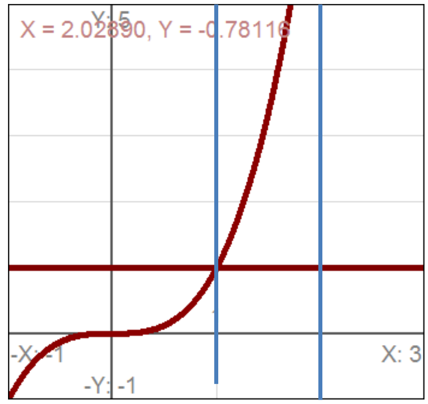
Surfaces of Revolution

Surface area problems in one variable are related to arc length in one variable. Recall our arc length formula is $s=\int_a^b \sqrt{1+\left[f'(x)\right]^2}\,dx$ where f(x) is the function we are measuring the length on. If we take this stretch of curve and revolve it around a graph, we get a surface. In the previous section we calculated the volume of the interior, here we are just interested in the surface area of the exterior. Our general formula for the surface area is $SA=2\pi\int_a^b r(x)\sqrt{1+\left[f'(x)\right]^2}\,dx$ or $SA=2\pi\int_a^b r(y)\sqrt{1+\left[f'(y)\right]^2}\,dy$ depending on the variable our function is in. The axis of rotation will change the radius function r(x) or r(y) but won't change anything else about the problem. In that way, surfaces are actually much easier than volumes.

- If you are rotating around the x-axis and integrating in x-variables, your r(x) function will just be the function f(x). If it's a line parallel to the x-axis then it will be function minus axis of rotation.
- If you are rotating around the y-axis and integrating in x-variable, your r(x) function will just be the variable x. If it's a line parallel to the y-axis, then it will be the variable minus the axis of rotation.
- If you are using y-variables and rotating around the y-axis, then use f(y) (or the function minus the axis of rotation) for r(y).
- If you are using y-variables and rotating around the x-axis, then use y (or the variable minus the axis of rotation).

Notice that the two cases are parallel. In one scenario you match the axis of rotation and you do one thing, and in the other your variable is opposite the axis of rotation and you do another. Compare this to the Washer and Shell Methods for volumes.

Example 5. Find the *surface area* of revolution for the curve $y = x^3$ revolved around the line x-axis on the interval [1,2].



As with the arc length, we can only integrate certain functions by hand here without advanced integration techniques, but we have a few more choices for surface areas than for arc length, because we may be able to do substitution. Here, our derivative is $y' = 3x^2$, so we get

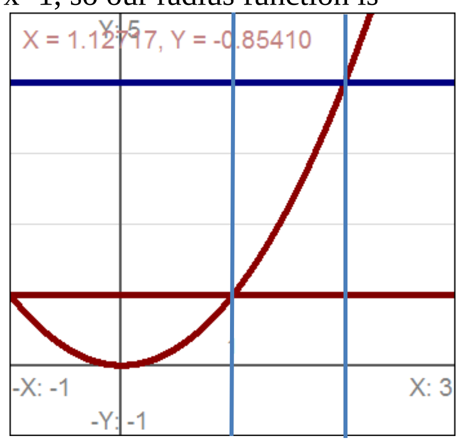
$SA = 2\pi\int_a^b r(x)\sqrt{1+\left[f'(x)\right]^2}\,dx = 2\pi\int_1^2 x^3\sqrt{1+9x^4}\,dx$. Because we are revolving around the x-axis, our function is r(x).

Substitution will yield and answer: $u = 1+9x^4, du = 36x^3dx \Rightarrow \frac{1}{36}du = x^3dx$.

$2\pi\int_1^2 x^3\sqrt{1+9x^4}\,dx = 2\pi\int_{10}^{145}\frac{1}{36}u^{1/2}du = \frac{\pi}{18}\cdot\frac{2}{3}\left[u^{3/2}\right]_{10}^{145} = \frac{\pi}{27}\left[145^{3/2} - 10^{3/2}\right]$. Like arc length problems, expect to routinely get very ugly numbers.

Example 6. Find the *surface area* of revolution for the curve $y = x^2$ revolved around the line x=1 on the interval [1,2].

Our derivative is just 2x, and we are revolving it around the line x=1, so our radius function is $r(x) = x-1$. So we get



$$SA = 2\pi\int_a^b r(x)\sqrt{1+\left[f'(x)\right]^2}\,dx = 2\pi\int_1^2 (x-1)\sqrt{1+2x^2}\,dx$$

$$2\pi\left[\int_1^2 x\sqrt{1+2x^2}\,dx - \int_1^2\sqrt{1+2x^2}\,dx\right]$$

The first integral we can use u-substitution on, but the second piece we will need trig substitution. Until we get to that section, evaluate it numerically in your calculator. In general, we will need to only revolve around a major axis (the x-axis or the y-axis) if we wish to avoid the especially nasty integrals. However, expect to be able to set up such problems even if you are not asked to integrate them by hand (yet).

Practice Problems.

6. For each of the curves below, find the surface area of revolution revolved around first the x-axis, and then the y-axis. Set up the equations. If they can be completed with u-substitution, finish them; if not, stop at the integral.
   a. $y = 2x+1$ on the interval [0,2]

b. $y = x^2$ on the interval [1,3]

c. $y = \frac{1}{2}\sin x$ on the interval [0,π/2]

d. $y = \cosh x$ on the interval [0,ln2]

e. $x = y^2 - 4$ on the interval [0,2]

f. $x = y^2 + y$ on the interval [0,1]

7. Set up the same problems to revolve around the line y= -2, and x=4. Don't integrate these, just set up the equations.