Betsy McCall

# Partial Fractions

Partial Fractions is an algebraic technique we employ in calculus to get a rational function into a form that conforms to one of our integration rules. The technique is sometimes taught in precalculus courses because it is only advanced algebra. The calculus doesn't come until the very last step.

Partial fractions are employed in specific circumstances:

a. The rational function is "proper", i.e. the degree of the numerator is strictly less than the degree of the denominator.
b. The rational function cannot be integrated as is by the use of inverse trig functions, substitution, or some combination of these.
c. The denominator is factorable.

It is this last condition that allows us to decompose the rational expression into simpler expressions that may yield more readily to our other techniques. If the rational function does not meet criteria (a), perform long division first, and then revisit the possible use of partial fractions if the remainder still meets criteria (b).

1. With linear factors, such as in $\frac{3x+1}{(x-2)(x+5)}$, we rewrite as an unknown constant over each linear factor separately: $\frac{A}{(x-2)}+\frac{B}{(x+5)}$.
2. With repeated linear factors, we use terms as above, but for the repeated factor, we have one with a linear denominator, one term with a squared denominator, one term with a cubic denominator, and so on, up to the multiplicity of the factor. In each case, the numerator remains a single unknown constant. For example: $\frac{3x+1}{(x-2)^3(x+5)}$ is rewritten as $\frac{A}{(x-2)}+\frac{B}{(x-2)^2}+\frac{C}{(x-2)^3}+\frac{D}{(x+5)}$. (Typically, you will only see multiplicity-2 problems that will need to be solved all the way, but you may be asked to set up more complicated ones.)
3. With quadratic, prime factors (i.e. with complex roots), we choose a linear numerator. Like the linear factors, it is one degree less than the degree of the denominator. For example: $\frac{3x+1}{(x^2+4)(x+5)}$ would be rewritten as $\frac{Ax+B}{x^2+4}+\frac{C}{x+5}$.
4. You could also encounter repeated quadratic factors: $\frac{3x+1}{(x^2+1)^2(x+5)}$ which would be rewritten as $\frac{Ax+B}{x^2+1}+\frac{Cx+D}{(x^2+1)^2}+\frac{E}{x+5}$. It's possible that you might have to set these up, but probably not solve it all the way.

I am sometimes asked why we set up the repeated factors the way we do. For instance, why can't you set up $\frac{3x+1}{(x-2)^2(x+5)}$ like $\frac{Ax+B}{(x-2)^2}+\frac{C}{x+5}$, or $\frac{3x+1}{(x-2)^3(x+5)}$ as $\frac{Ax^2+Bx+C}{(x-2)^3}+\frac{D}{x+5}$, and well, the truth is, you *could*, BUT the result may not be in a nice form that can be integrated easily, which is our ultimate goal. The terms like $\frac{A}{(x-2)}$ can potentially be integrated through the log rule, as can $\frac{Bx}{x^2+4}$ if we split the first term off from $\frac{Bx+C}{x^2+4}$; and the $\frac{C}{x^2+4}$ that is leftover is a straightforward inverse tangent rule. However, a more complicated denominator, like $x^2+x+1$, which is also not factorable (though it may or may not have real solutions), might require both terms of the numerator we found to integrate by substitution.) Terms like $\frac{A}{(x-2)^2}$ can be integrated by the generalized power rule, something that isn't possible for $\frac{Bx+C}{(x-2)^2}$. Algebra might be able to get it into a workable form, but that would be still more work. So, part of this process is not just to break down the rational expression into smaller parts that are equivalent to the original, but also to get them into forms that are ***easily integrated***.

Let's consider two examples.

A. $\frac{x^2+4}{(x-2)^2(x+5)}$. Decompose this by means of partial fractions.

First, we rewrite with our three unknowns: $\frac{A}{(x-2)^2}+\frac{B}{x-2}+\frac{C}{x+5}$. To solve for the three variables, we need to write the three fractions with a common denominator and add. $\frac{A}{(x-2)^2}\frac{(x+5)}{(x+5)}+\frac{B}{x-2}\frac{(x-2)(x+5)}{(x-2)(x+5)}+\frac{C}{x+5}\frac{(x-2)^2}{(x-2)^2}$

$$\frac{A(x+5)+B(x^2+3x-10)+C(x^2-4x+4)}{(x-2)^2(x+5)}=\frac{Ax+5A+Bx^2+3Bx-10B+Cx^2-4Cx+4C}{(x-2)^2(x+5)}$$

Now that the denominators are the same, the numerators must be also, i.e. $Ax+5A+Bx^2+3Bx-10B+Cx^2-4Cx+4C=x^2+4$ or without distributing $A(x+5)+B(x-2)(x+5)+C(x-2)^2=x^2+4$. We will return to this later because there are two techniques for finding the constants A,B,C.

1. Using the expanded form $Ax+5A+Bx^2+3Bx-10B+Cx^2-4Cx+4C=x^2+4$, we will collect coefficients on both sides for each degree term to create an equation. In other words, since the coefficient of $x^2$ on the right is 1, the coefficients of the $x^2$ terms on the left have to add up to 1 as well: $Bx^2+Cx^2=x^2$ or $B+C=1$ We should do likewise for the linear terms and the constants. If a term is missing (here on the right), then the coefficient for that term is 0. This gives us: $Ax+3Bx-4Cx=0$ or $A+3B-4C=0$, and $5A-10B+4C=4$.

This process creates a three-variable system of equations to solve: $\begin{cases} B+C=1 \\ A+3B-4C=0 \\ 5A-10B+4C=4 \end{cases}$.

Use whatever techniques are available to you, including substitution, elimination, (graphing, if you have only two variables), or matrix methods (if you know them).

By substitution, C=1-B, which leaves us: $\begin{cases} A+3B-4(1-B)=0 \\ 5A-10B+4(1-B)=4 \end{cases} \rightarrow \begin{cases} A+7B=4 \\ 5A-14B=0 \end{cases}$.

Multiply the top equation by 2 and add 7A= 8, thus $A=\frac{8}{7}$, and backsolve from there to get the others, $B=\frac{20}{49}$, and $C=\frac{29}{49}$.

Solutions are frequently unpleasant numbers. It takes a lot of effort to make them nice, and you rarely encounter nice problems in real applications.

2. The second method involves preserving the factors, as in $A(x+5)+B(x-2)(x+5)+C(x-2)^2=x^2+4$, and substitute selective values of x to reduce the equations. For instance, suppose I choose x=2. All the factors containing (x-2) reduce to zero, and eliminates those terms entirely. Here, this gives: $A(2+5)+B(2-2)(2+5)+C(2-2)^2=2^2+4 \rightarrow A(7)=8$, thus, as before $A=\frac{8}{7}$. We can similarly choose (-5) to eliminate the other factor. $A(-5+5)+B(-5-2)(-5+5)+C(-5-2)^2=(-5)^2+4$ $\rightarrow C(-7)^2=25+4 \rightarrow 49C=29$, and thus $C=\frac{29}{49}$. If you still have variables unsolved for (and you will with repeated or quadratic factors), choose another number like zero to create an equation that you can use to backsolve with the other two variables. I'll use zero: $A(0+5)+B(0-2)(0+5)+C(0-2)^2=0^2+4 \rightarrow A(5)+B(-10)+C4=4$ (which was our third equation from above.

   Many students prefer the first method because it is familiar to them, but as you can see, both methods produce equivalent results.

   Our final decomposition is: $\frac{8/7}{(x-2)^2}+\frac{20/49}{x-2}+\frac{29/49}{x+5}$.

B. $\frac{4x-7}{(x^2+1)(x-3)}$ Let's try this one as our second example. $\frac{Ax+B}{x^2+1}+\frac{C}{x-3}$. Finding a common denominator yields: $\frac{(Ax+B)(x-3)+C(x^2+1)}{(x^2+1)(x-3)}$ or that $(Ax+B)(x-3)+C(x^2+1)=4x-7$.

1. Using method 1: $Ax^2-3Ax+Bx-3B+Cx^2+C=4x-7$, and from that, we get that for the $x^2$ terms: $A+C=0$, for the x terms: $-3A+B=4$, and for the constants: $-3B+C=-7$. If we let C= -A, the third equation becomes: $-A-3B=-7$, and if we multiply the second equation by three and add to this one we get: $-10A=5$ or $A=-\frac{1}{2}$. Thus, $C=\frac{1}{2}$, and $B=\frac{5}{2}$.

2. Using the second method on $(Ax+B)(x-3)+C(x^2+1)=4x-7$, let's start by letting x=3. $(3A+B)(3-3)+C(3^2+1)=4\cdot 3-7\rightarrow C10=5$ or $C=\frac{1}{2}$. Since we can't make any other factors 0, we just have to pick values of x. Let's try x=0. $(0\cdot A+B)(0-3)+C(0^2+1)=4\cdot 0-7\rightarrow(B)(-3)+C=-7$. Replacing C, we get: $\rightarrow(B)(-3)+\frac{1}{2}=-7\rightarrow -3B=-\frac{15}{2}$, or $B=\frac{5}{2}$. And then another random value for x, say, x=1. $(1A+B)(1-3)+C(1^2+1)=4\cdot 1-7\rightarrow(1A+B)(-2)+C(2)=-3$ $\rightarrow -2A-(2)\frac{5}{2}+\frac{1}{2}(2)=-3\rightarrow -2A-5+1=-3\rightarrow -2A=1$ or $A=-\frac{1}{2}$.

So, our final decomposition is $\frac{-\frac{1}{2}x+\frac{5}{2}}{x^2+1}+\frac{\frac{1}{2}}{x-3}$. And we would integrate this as three terms: $\left(-\frac{1}{2}\right)\frac{x}{x^2+1}+\left(\frac{5}{2}\right)\frac{1}{x^2+1}+\left(\frac{1}{2}\right)\frac{1}{x-3}$.

**Problems.** Decompose the following rational expressions using partial fractions. (Beware: you may need to do long division first!) Solve for the variables unless otherwise indicated. Integrate where indicated.

i. $\frac{3x+1}{(x-2)(x+5)}$

ii. $\frac{x^2+11x}{x^2+5x+6}$

iii. $\frac{1}{x^3-1}$

iv. $\dfrac{x^2-6}{(x+1)(x+4)(x-3)}$

v. $\dfrac{2x+3}{(x-1)^2}$

vi. $\dfrac{x^2+2x+4}{(x-3)^2(x+1)}$

vii. $\dfrac{x^3+4x+1}{(x-2)^3(x+7)}$ (set up only)

viii. $\dfrac{4x+13}{(x^2+4)(x-3)^2}$

ix. $\dfrac{x^2+3x-10}{(x^2+2)^2(x-9)}$ (set up only)

x. $\dfrac{x^3+5x^2-8x-24}{(x^2+1)^2(x-3)^3(x+4)(x-1)}$ (set up only)

xi. $\displaystyle\int \frac{5x-12}{x^2-4}\,dx$

xii. $\displaystyle\int \frac{2x-3}{(x^2+16)(x-2)}\,dx$

xiii. $\displaystyle\int \frac{10x^2-x-17}{x^2(x+3)(x-1)}\,dx$