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Integration by Parts

Integration by parts is used when two functions are multiplied together, but which are not related to each other through a chain rule (when substitution is used). Integration by parts is essentially in inverse operation on the product rule when taking derivatives, and that's where we get the formula $\int udv = uv - \int vdu$.

Integration by parts works best when the function we choose as *u* gets simpler when we take its derivative, and/or the function we choose as *dv* gets simpler when we integrate it. To ensure that, there is a basic rule of thumb we can use when choosing u (or we can choose dv using the reverse rule).

Choice	u	dv
1st	Logs: $\ln(x), \log_{10}(x)$	Exponential functions: $e^x, 2^x$, etc.
2nd	Inverse trig functions: $\sin^{-1}(x), \arctan(x)$, etc.	Trig functions: $\sin(x), \tan(x)$, etc.
3rd	Algebraic functions (positive integer powers)	Algebraic function (negative integer powers)
4th	Algebraic function (other)	Algebraic function (other)
5th	Trig functions: $\sin(x), \tan(x)$, etc.	*
6th	Exponential functions: $e^x, 2^x$, etc.	*

* These functions can't be integrated by themselves. Never use inverse trig function or logs as *dv*.

You can use **LIATE** to remember the sequence of function for u: **L**ogs, **I**nverse trig, **A**lgebraic, **T**rig, **E**xponential.

Not everything can be integrated by parts. Case in point: $\int \ln(x)\arcsin(x)dx$. Neither of these functions can integrated alone, so neither can be dv.

Let's see some examples.

1. Logs and inverse trig functions by themselves can be integrated by parts.

   $\int \ln(x)dx$

   Choose $u = \ln x, dv = dx$. Find $du = \frac{1}{x}dx, v = x$

   $\int \ln(x)dx = x\ln x - \int x\frac{1}{x}dx = x\ln x - \int dx = x\ln x - x + C$

2. $\int x\arctan(x)dx$

   Choose $u = \arctan(x), dv = xdx$. Find $du = \frac{1}{x^2+1}dx, v = \frac{1}{2}x^2$

   $\int x\arctan(x)dx = \frac{1}{2}x^2\arctan(x) - \frac{1}{2}\int \frac{x^2}{x^2+1}dx$. This new integral requires long division.

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$$\frac{1}{2}x^2\arctan(x)-\frac{1}{2}\int 1-\frac{1}{x^2+1}dx=\frac{1}{2}x^2\arctan(x)-\frac{1}{2}\left[x-\arctan(x)\right]+C$$
$$=\frac{1}{2}\left[x^2\arctan(x)-x+\arctan(x)\right]+C$$

3. $\int x\sqrt{x+1}dx$

This integral can be done by substitution as well (with $u=\sqrt{x+1}$), but we will do it by parts.

Choose $u=x, dv=\left(x+1\right)^{1/2}dx$. Find $du=dx, v=\frac{2}{3}(x+1)^{3/2}$

$$\int x\sqrt{x+1}dx=\frac{2}{3}x(x+1)^{3/2}-\frac{2}{3}\int (x+1)^{3/2}dx=\frac{2}{3}x(x+1)^{3/2}-\frac{2}{3}\left[\frac{2}{5}(x+1)^{5/2}\right]+C$$

4. $\int \frac{x}{(x+3)^2}dx$

Choose $u=x, dv=(x+3)^{-2}dx$. Find $du=dx, v=-(x+3)^{-1}=-\frac{1}{x+3}$

$$\int \frac{x}{(x+3)^2}dx=-\frac{x}{x+3}+\int\frac{1}{x+3}dx=-\frac{x}{x+3}+\ln(x+3)+C$$

5. $\int x^3e^{x^2}dx$

Choose $u=x^2, dv=xe^{x^2}dx$. We do this because we will need to integrate dv by substitution. The exponential function here will not integrate without the x. Find $du=2xdx, v=\frac{1}{2}e^{x^2}$.

$$\int x^3e^{x^2}dx=\frac{1}{2}x^2e^{x^2}-\int\frac{1}{2}2xe^{x^2}dx=\frac{1}{2}x^2e^{x^2}-\int xe^{x^2}dx=\frac{1}{2}x^2e^{x^2}-\frac{1}{2}e^{x^2}+C$$

For some integrals, we will have to use by parts more than once.

6. $\int \sin(x)e^x dx$

Choose $u=\sin(x), dv=e^x dx$. Find $du=\cos(x)dx, v=e^x$

$$\int\sin(x)e^xdx=\sin(x)e^x-\int\cos(x)e^xdx$$

When you do the next step, be consistent in choosing *u* and *dv*.

Choose $u=\cos(x), dv=e^x$. Find $du=-\sin(x)dx, v=e^x$

$$\int\sin(x)e^xdx=\sin(x)e^x-\int\cos(x)e^xdx=\sin(x)e^x-\left[\cos(x)e^x+\int\sin(x)e^xdx\right]$$
$$\int\sin(x)e^xdx=\sin(x)e^x-\cos(x)e^x-\int\sin(x)e^xdx$$

This is a special kind of integral. We have here the same integral we started with, so in this case add it to the left side to finish solving.

$2\int \sin(x)e^x dx = \sin(x)e^x - \cos(x)e^x + C$, so our final answer we get from dividing by 2.

$$\int \sin(x)e^x dx = \frac{1}{2}\sin(x)e^x - \frac{1}{2}\cos(x)e^x + C$$

Check your answer by differentiating using the product rule.

7. $\int x^2 e^x dx$

Choose $u = x^2, dv = e^x dx$. Find $du = 2xdx, v = e^x$.

$$\int x^2 e^x dx = x^2 e^x - 2\int xe^x dx$$

For the new integral, choose $u = x, dv = e^x dx$, find $du = dx, v = e^x$. & integrate again.

$$\int x^2 e^x dx = x^2 e^x - 2\int xe^x dx = x^2 e^x - 2\left[xe^x - \int e^x dx\right]$$

Apply by parts until you have an integral that can be integrated by substitution or a basic rule.

$$\int x^2 e^x dx = x^2 e^x - 2xe^x + 2e^x + C$$

For very long by parts problems, where we will have to do integration by parts repeatedly, there is the tabular method to generate the terms we need.

8. $\int x^5 e^{2x} dx$

Choose $u = x^5, dv = e^{2x}dx$. Then create a table. Differentiate in the u column until you can't any more. And then in the dv column, integrate. In the first column, keep track of the signs.

Sign	**u**	**dv**
+	x^5	e^{2x}
-	$5x^4$	$\frac{1}{2}e^{2x}$
+	$20x^3$	$\frac{1}{4}e^{2x}$
-	$60x^2$	$\frac{1}{8}e^{2x}$
+	$120x$	$\frac{1}{16}e^{2x}$
-	120	$\frac{1}{32}e^{2x}$
+	0	$\frac{1}{64}e^{2x}$

Combine terms following the arrows:

$$\int x^5 e^{2x} dx = \frac{1}{2}x^5 e^{2x} - \frac{5}{4}x^4 e^{2x} + \frac{5}{2}x^3 e^{2x} - \frac{15}{4}x^2 e^{2x} + \frac{15}{4}xe^{2x} - \frac{15}{8}e^{2x} + C$$

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9. Some problems defy the general principles, and can only be done by trial and error. Case in point: $\int \frac{xe^{2x}}{(2x+1)^2}dx$

   You can try choosing according to the LIATE rule, and you will get nowhere. (Try it.) It just gets ugly. But there is a choice that does work: $u = xe^{2x}, dv = (2x+1)^{-2}dx$. Our choose for u will have to be differentiated using the product rule. $du = 2xe^{2x} + e^{2x} = e^{2x}(2x+1), v = -\frac{1}{2}\cdot\frac{1}{2x+1}$. Putting these into our formula:

   $$\int \frac{xe^{2x}}{(2x+1)^2}dx = -\frac{xe^{2x}}{2(2x+1)} + \frac{1}{2}\int \frac{2x+1}{2x+1}e^{2x}dx = -\frac{xe^{2x}}{2(2x+1)} + \frac{1}{2}\int e^{2x}dx = -\frac{xe^{2x}}{2(2x+1)} + \frac{1}{4}e^{2x} + C$$

Problems.

a. Choose *u* and *dv*. No need to integrate.
   - i. $\int \operatorname{arcsec}(x)dx$
   - ii. $\int x\ln(x)dx$
   - iii. $\int x^3\sin(x^2)dx$
   - iv. $\int \frac{x}{\sqrt{3x+4}}dx$
   - v. $\int \cos(x)e^{4x}dx$

b. Integrate.
   - vi. $\int \frac{x^2}{\sqrt[3]{2x+1}}dx$
   - vii. $\int x^2e^{-3x}dx$
   - viii. $\int x^2\arcsin(x)dx$
   - ix. $\int x\sec^2(2x)dx$
   - x. $\int \frac{xe^x}{(x+1)^2}dx$
   - xi. $\int \cos(3x)e^{-4x}dx$
   - xii. $\int (2x-1)^4\cos(5x)dx$

c. Give five examples of integrals that cannot (or should not) be done with by parts.