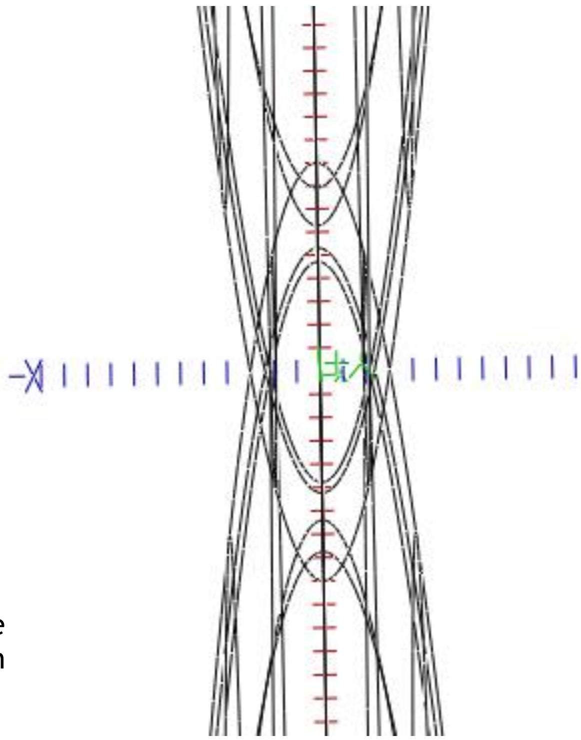
Triple Integrals

The most difficult thing about setting up triple integrals is visualizing what is going on. Part of the reason for this is because we aren't used to thinking of graphs in three dimensions. They are hard to draw. One of the techniques we will consider is thinking of the graphs in two dimensions, a slice of the full graph, to orient us in for the first set of limits, and then our equations will reduce naturally to two dimensions, and then we will be back on more familiar territory. Another reason for the difficulty is that we may be working in coordinates we aren't that familiar with, such as cylindrical or spherical. It takes practice to get good at these, but at some level, we can treat some of it algebraically.

We work with these different systems of coordinates because they may make the integration simpler, or, at the very least, doable in one integral, as opposed to several integrals. I will do no integrating here, we will only be setting up integrals. But to be sure that you are doing it correctly, you should be able to get the same answer out of every version, though you may need math software to verify this without a great deal of work for some problems.

Example 1. Find the volume bounded by the graphs of $z=9-x^2-y^2$ and $z=x^2+y^2-9$.



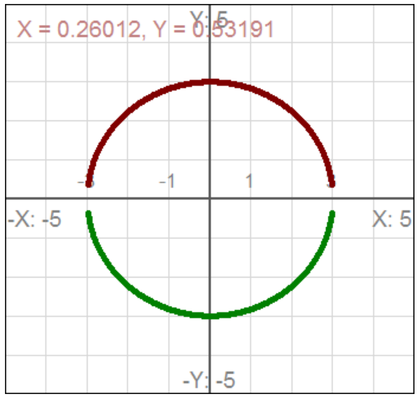
The graph of the two functions is shown here in wireframe. They are two intersecting paraboloids. Let's begin with rectangular coordinates.

The limits in z are just the two functions. They are both pointing basically vertically, and there is no change of top and bottom functions anywhere, unlike our last example.

To get the x-y equations and limits, we need to set the two z equations equal to each other.

$$9-x^2-y^2=x^2+y^2-9$$
$$2x^2+2y^2=18 \Rightarrow x^2+y^2=9$$
$$y=\pm\sqrt{9-x^2}$$

In the x-y plane, the projection of our region looks like a circle of radius 3, centered at the origin. Our limits in y are given above, and the limits in x are then -3 to 3.



Our integral then is:

$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{x^2+y^2-9}^{9-x^2-y^2} dzdydx$$

This isn't too horrible, but it will probably take trig substitution to finish integrating. Compared to the first example, though, this is a piece of cake.

Let's try this in cylindrical coordinates.

First, rewrite the equations for z in cylindrical. Then solve for r.

$$z = 9 - x^2 - y^2 \Rightarrow z = 9 - r^2$$
$$z = x^2 + y^2 - 9 \Rightarrow z = r^2 - 9$$
$$9 - r^2 = r^2 - 9$$
$$2r^2 = 18 \Rightarrow r^2 = 9 \Rightarrow r = 3$$

It's not difficult, as shown to get limits for r as 0 and 3. And the limits for θ are 0 and 2π since we want to go all the way around the circle.

Our integral then ends up being:

$$\int_0^{2\pi}\int_0^3\int_{r^2-9}^{9-r^2} r\,dz\,dr\,d\theta$$

Spherical turns out to be the really hard one. Doing the conversion:

$$z = 9 - x^2 - y^2 \Rightarrow \rho\cos\varphi = 9 - \left(\rho^2\cos^2\theta\sin^2\varphi + \rho^2\sin^2\theta\sin^2\varphi\right) = 9 - \rho^2\sin^2\varphi$$
$$z = x^2 + y^2 - 9 \Rightarrow \rho\cos\varphi = \rho^2\cos^2\theta\sin^2\varphi + \rho^2\sin^2\theta\sin^2\varphi - 9 = \rho^2\sin^2\varphi - 9$$

We need to solve for ρ.

$$\rho\cos\varphi = 9 - \rho^2\sin^2\varphi \Rightarrow \rho^2 + \rho\cot\varphi\csc\varphi = 9\csc^2\varphi$$
$$\rho^2 + \rho\cot\varphi\csc\varphi + \left(\frac{1}{2}\cot\varphi\csc\varphi\right)^2 = 9\csc^2\varphi + \left(\frac{1}{2}\cot\varphi\csc\varphi\right)^2$$
$$\left(\rho + \frac{1}{2}\cot\varphi\csc\varphi\right)^2 = \csc^2\varphi\left(9 + \frac{1}{4}\cot^2\varphi\right)$$
$$\rho = \csc\varphi\sqrt{9 + \frac{1}{4}\cot^2\varphi} - \frac{1}{2}\cot\varphi\csc\varphi$$
$$\rho\cos\varphi = \rho^2\sin^2\varphi - 9 \Rightarrow \rho = \csc\varphi\sqrt{9 + \frac{1}{4}\cot^2\varphi} + \frac{1}{2}\cot\varphi\csc\varphi$$

Here, we are going to have to set up two separate integrals. One for φ from 0 to π/2 (remember, this is the level of the x-y plane), and one for π/2 to π. However, because the graph is symmetrical above and below the axes, we can just do the top half of the graph, and double it. The solutions for ρ give away this symmetry. And of course, θ goes from 0 to 2π because we go all the way around.

Our final integral then is:

$$2\int_0^{2\pi}\int_0^{\pi/2}\int_0^{\csc\varphi\sqrt{9+\frac{1}{4}\cot^2\varphi}-\frac{1}{2}\cot\varphi\csc\varphi} \rho^2 \sin\varphi d\rho d\varphi d\theta$$

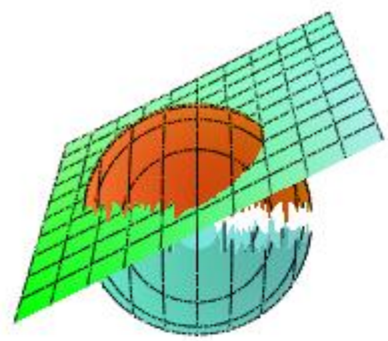
This is better than what we saw on the first example, but still not great. The square root term is the one that could be trickiest to integrate.

In this example, there is one clear best choice: cylindrical coordinates has nothing but constants and polynomials in it. Integrating this gives 81π.

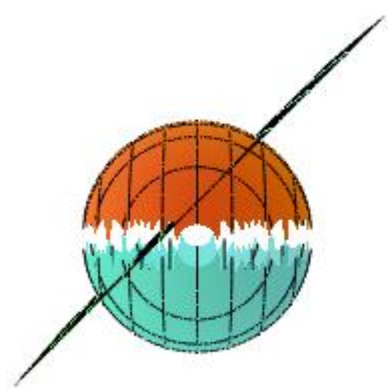
Let's do one more difficult example to drive home the steps.

Example 2. Find the volume of the solid bounded by the graph of $x^2+y^2+z^2=9$ and the graph of $x+y+2z=4$.

Let's consider the graph of what we are looking at.



This image is useful for us because the top half of the sphere and the bottom half of the sphere are coloured differently. Fortunately, our plane only cuts through the top half of the sphere. In general, planes may cut through part of one half and part of the other. In such cases, we'd have to split the integral in two places: the portion bounded by the plane and sphere, and a second part bounded by just the sphere. Compare with the second graph formed by the same sphere, and the plane x+z=1. For some part of this graph the "top function" is the top half of the sphere, and the "bottom function" is the bottom half of the sphere. For another part of the graph, the two functions are the top half of the sphere and the plane. To set this up in rectangular coordinates, we are going to need two integrals, at least in rectangular coordinates, unless we are a bit clever. But this is a bit messy; and frequently triple integrals are unless the functions are very cleverly chosen. The original equations are chosen so that we will just have the one integral. Find the limits for z by solving both equations for z.



$$\iint_R \int_{2-\frac{1}{2}x-\frac{1}{2}y}^{\sqrt{9-x^2-y^2}} dz dA$$

We don't have to do z first, but it's not likely another choice of initial variables will help much.

To get the next set of limits, we need to find the intersections of the graphs in x and y. Do a substitution to reduce the equations to a single equation in two variables.

$$z = 2 - \frac{1}{2}x - \frac{1}{2}y$$

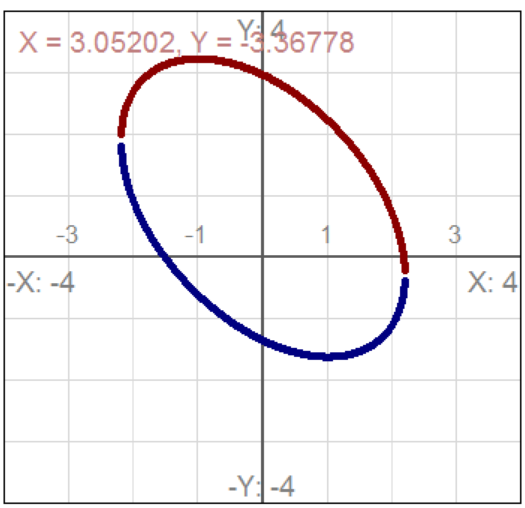
$$x^2 + y^2 + \left(2 - \frac{1}{2}x - \frac{1}{2}y\right)^2 = 9$$

$$x^2 + y^2 + 4 - x - y - x + \frac{1}{4}x^2 + \frac{1}{2}xy - y + \frac{1}{4}xy + \frac{1}{4}y^2 = 9$$

$$\frac{5}{4}x^2 + \frac{5}{4}y^2 - 2x - 2y + \frac{1}{2}xy = 5$$

$$x^2 + y^2 - \frac{8}{5}x - \frac{8}{5}y + \frac{2}{5}xy = 4$$

We need to choose one of the variables to solve for. I will choose y, but neither is going to be particularly pretty. Treat x like a "constant". You are going to need to complete the square. Not pretty.



$$y^2 + \left(\frac{2}{5}x - \frac{8}{5}\right)y = -x^2 + \frac{8}{5}x + 4$$

$$y^2 + \left(\frac{2}{5}x - \frac{8}{5}\right)y + \left[\frac{\left(\frac{2}{5}x - \frac{8}{5}\right)}{2}\right]^2 = -x^2 + \frac{8}{5}x + 4 + \left[\frac{1}{5}x - \frac{4}{5}\right]^2$$

$$\left(y + \frac{1}{5}x - \frac{4}{5}\right)^2 = -x^2 + \frac{8}{5}x + 4 + \frac{x^2 - 40x + 16}{25}$$

$$y = \pm\frac{1}{5}\sqrt{116 - 24x^2} - \left(\frac{1}{5}x - \frac{4}{5}\right)$$

This looks atrocious, but I can graph it now.

It's an ellipse, a rotated ellipse, but an ellipse.

These equations, as ugly as they are, are the limits in the y-direction. These equations are only defined when the square root is defined. We can solve for $116-24x^2=0$ to get our limits in x. This leaves us with the final pair of integrals.

$$\iint_R \int_{-\sqrt{9-x^2-y^2}}^{1-x-y} dz\,dA$$

$$\int_{-3/\sqrt{2}}^{3/\sqrt{2}} \int_{-\frac{1}{5}\sqrt{116-24x^2}-\left(\frac{1}{5}x-\frac{4}{5}\right)}^{\frac{1}{5}\sqrt{116-24x^2}-\left(\frac{1}{5}x-\frac{4}{5}\right)} \int_{-\sqrt{9-x^2-y^2}}^{1-x-y} dz\,dy\,dx$$

Do I want to actually integrate this mess? No, not especially. When they get this complicated, that's what computers are for.

Let's see if we can choose a different set of coordinates and maybe get something a bit simpler.

Next, try cylindrical.

Because the z variable doesn't change in cylindrical relative to rectangular, we are still going to be faced with the problem of having two integrals, or calculating what we don't need and subtracting it off the total. What will change is that our expressions for z will be functions of r and θ instead of x and y.

$$r^2+z^2=9$$

$$z=2-\frac{1}{2}r\cos\theta-\frac{1}{2}r\sin\theta$$

So our two integrals become:

$$\iint_R \int_{-\sqrt{9-r^2}}^{1-\frac{1}{2}r\cos\theta-\frac{1}{2}r\sin\theta} dz\,dA$$

As before, substitute one equation into the other and then solve for r.

$$r^2+\left(2-\frac{1}{2}r\cos\theta-\frac{1}{2}r\sin\theta\right)^2=9$$

$$r^2+4-r\cos\theta-r\sin\theta-r\cos\theta+\frac{1}{4}r^2\cos^2\theta+\frac{1}{4}r^2\sin\theta\cos\theta-r\sin\theta+\frac{1}{4}r^2\sin^2\theta+\frac{1}{4}r^2\sin\theta\cos\theta=9$$

$$\frac{5}{4}r^2+\frac{1}{2}r^2\cos\theta\sin\theta-2r(\cos\theta+\sin\theta)=5$$

$$(5+2\cos\theta\sin\theta)r^2-8r(\cos\theta+\sin\theta)=20$$

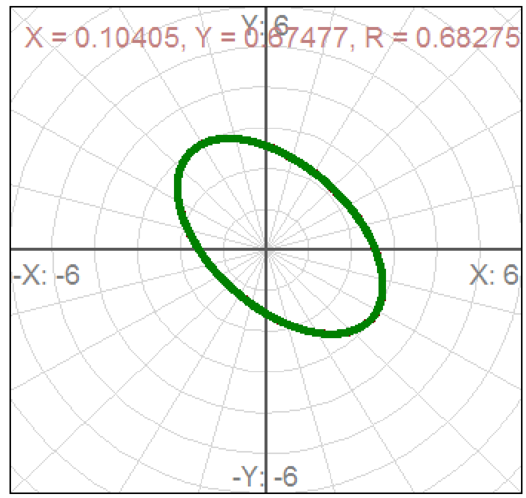
$$r^2-\frac{8r(\cos\theta+\sin\theta)}{5+2\cos\theta\sin\theta}=\frac{20}{5+2\cos\theta\sin\theta}$$

$$r^2-\frac{2(4\cos\theta+4\sin\theta)}{5+2\cos\theta\sin\theta}+\left(\frac{4\cos\theta+4\sin\theta}{5+2\cos\theta\sin\theta}\right)^2=\frac{20(5+2\cos\theta\sin\theta)}{(5+2\cos\theta\sin\theta)^2}+\left(\frac{4\cos\theta+4\sin\theta}{5+2\cos\theta\sin\theta}\right)^2$$

$$\left[r-\frac{4\cos\theta+4\sin\theta}{5+2\cos\theta\sin\theta}\right]^2=\frac{100+40\cos\theta\sin\theta+16\cos^2\theta+16\sin^2\theta+32\cos\theta\sin\theta}{(5+2\cos\theta\sin\theta)^2}$$

$$\left[r-\frac{4\cos\theta+4\sin\theta}{5+2\cos\theta\sin\theta}\right]^2=\frac{116+62\cos\theta\sin\theta}{(5+2\cos\theta\sin\theta)^2}$$

$$r=\frac{\pm\sqrt{116+62\cos\theta\sin\theta}+4\cos\theta+4\sin\theta}{5+2\cos\theta\sin\theta}$$



We only need one version because they both graph the same. And surprise, it's still one of those tilted ellipses.

The limits in θ are just 0 to 2π. So, we end up with this:

$$\int_0^{2\pi}\int_0^{\frac{\sqrt{116+62\cos\theta\sin\theta}+4\cos\theta+4\sin\theta}{5+2\cos\theta\sin\theta}}\int_{-\sqrt{9-r^2}}^{2-\frac{1}{2}r\cos\theta-\frac{1}{2}r\sin\theta} r\,dz\,dr\,d\theta$$

Well, that was even more atrocious. Let's try spherical. We may have to try the same trick with spherical, since our plane cuts below the center, we will depend on ρ being negative for some parts of the calculation. If we don't get the same answer as before, we can subtract the result from the entire sphere as we did before; we don't even have to change the limits. Therefore, I will only set up the one integral.

$$\rho=3$$

$$\rho\sin\varphi\cos\theta+\rho\sin\varphi\sin\theta+2\rho\cos\varphi=4$$

These are our original equations in spherical coordinates. We need to solve the second equation for ρ.

$$\rho\sin\varphi\cos\theta+\rho\sin\varphi\sin\theta+2\rho\cos\varphi=4$$
$$\rho=\frac{4}{\sin\varphi(\cos\theta+\sin\theta)+2\cos\varphi}$$

There's no particular reason why we should start with ρ, but let's do it anyway since it's the easiest to solve for.

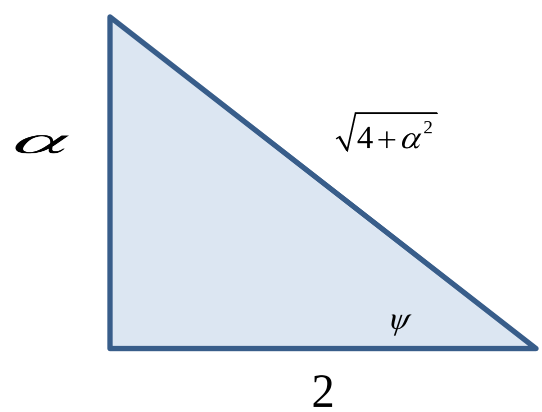
$$\iint_R \int_{\frac{4}{\sin\varphi(\cos\theta+\sin\theta)+2\cos\varphi}}^{3} \rho^2\sin\varphi\, d\rho\, dA$$

To get the next set of limits, we need to set the two ρ equations equal to each other and solve for, let's say, φ.

$$\sin\varphi(\cos\theta+\sin\theta)+2\cos\varphi=\frac{1}{3}$$

To do this, we are going to need some algebra tricks. Let's set $\alpha=\cos\theta+\sin\theta$. Our equation is now $\alpha\sin\varphi+2\cos\varphi=\frac{1}{3}$. The trick we are going to do is divide the equation by $\sqrt{4+\alpha^2}$. We then get:

$$\frac{\alpha}{\sqrt{4+\alpha^2}}\sin\varphi+\frac{2}{\sqrt{4+\alpha^2}}\cos\varphi=\frac{1}{3\sqrt{4+\alpha^2}}$$



Consider the triangle shown at right.

We are going to treat these coefficients as though they are the trig functions of this dummy angle ψ.

On the right side, remember, it's just a constant, but the left becomes.

$$\sin\psi\sin\varphi+\cos\psi\cos\varphi=\frac{1}{3\sqrt{4+\alpha^2}}$$
$$\cos(\varphi-\psi)=\frac{1}{3\sqrt{4+\alpha^2}}$$

Solving for φ now:

$$\cos(\varphi-\psi)=\frac{1}{3\sqrt{4+\alpha^2}}$$

$$\varphi-\psi=\cos^{-1}\left(\frac{1}{3\sqrt{+\alpha^2}}\right)\Rightarrow\varphi=\cos^{-1}\left(\frac{1}{3\sqrt{+\alpha^2}}\right)+\psi$$

$$\psi=\cos^{-1}\left(\frac{1}{\sqrt{4+\alpha^2}}\right)$$

Putting α back to θs.

$$\varphi=\cos^{-1}\left(\frac{1}{3\sqrt{4+(\cos\theta+\sin\theta)^2}}\right)+\cos^{-1}\left(\frac{1}{\sqrt{4+(\cos\theta+\sin\theta)^2}}\right)$$

$$\varphi=\cos^{-1}\left(\frac{1}{3\sqrt{5+2\sin\theta\cos\theta}}\right)+\cos^{-1}\left(\frac{1}{\sqrt{5+2\cos\theta\sin\theta}}\right)$$

We start at the z-axis as φ=0, so this is just the upper limit. And just to be clear, you can't add these any more than you can add cosines with different angles.

The limits in θ are still 0 to 2π.

The final version of the triple integral is:

$$\int_0^{2\pi}\int_0^{\cos^{-1}\left(\frac{1}{3\sqrt{5+2\sin\theta\cos\theta}}\right)+\cos^{-1}\left(\frac{1}{\sqrt{5+2\cos\theta\sin\theta}}\right)}\int_{\frac{1}{\sin\varphi(\cos\theta+\sin\theta)+\cos\varphi}}^{3}\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$

Did I say this was messy? This is a much harder problem than you will normally encounter, but it's good in that it reveals a lot of little algebra tricks you can use to solve problems. It also highlights the fact that in many problems you will be asked to set up the integral, but not necessarily to actually doing the integrating if it's especially nasty. One hopes to find problems that have enough symmetry in some coordinate system so that they reduce to something simple.

Practice Problems. Find the volume bounded by the graphs of the equations. Set up the integrals in all three coordinate systems. If you get a really easy one, as in Example 1, integrate it.

1. $z=y^2, x=3, x=-3, 4y+12z=19$
2. $x=-4, x=4, z=-10, z=y^2-x^2, y=-1, y=1$
3. $x^2+y^2-z^2=1, z=-2, z=4$
4. $z=\sqrt{x^2+y^2}, x+y+2z=6$
5. $x^2+z^2=16, y=9-x^2-z^2, y=0$