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Tangents & Normals

We encounter problems involving tangents and normals several times throughout the course, so we will return to this handout a couple different times. In some cases we will be looking at tangent vectors (along a curve) together with a normal vector, and the binormal vector (together, they define a three-dimensional space). In later cases we will be considering tangent planes (to a surface), and normal vectors; in one case the surface will be defined using traditional coordinate systems (rectangular, cylindrical or spherical), and in the second case, the surface will be defined parametrically. Each of these cases is a bit different, but they also have certain similarities, such as the terminology, and can be easily confused, especially once we have to consider all of them.

Parametric Curves in 3-space.
Given a parameterized curve, we want to know what the tangent vector to the curve is, what direction is it pointing in at any given instant along the path. Finding this vector is also useful for setting up a coordinate system that changes as the particle moved along the path.

Example 1. Consider the parameterized curve $\vec{r}(t)=3t\vec{i}-t\vec{j}+t^2\vec{k}$, find the unit tangent vector to the curve, the unit normal to the curve, and the unit binormal.

The first step is to find $\vec{r}\,'(t)=3\vec{i}-\vec{j}+2t\vec{k}$. This vector gives us both the direction, and a magnitude. However, we want an unit tangent vector, so we need to calculate the length of this vector and divide by it.

$$\vec{T}(t)=\frac{\vec{r}\,'(t)}{\left\|\vec{r}\,'(t)\right\|}=\frac{3\vec{i}-\vec{j}+2t\vec{k}}{\sqrt{10+4t^2}}$$

The first step isn't so bad, but the second step, for a problem like this one can be problematic because the unit normal is derived from the unit tangent vector, not just the acceleration vector.

$$\vec{N}(t)=\frac{\vec{T}\,'(t)}{\left\|\vec{T}\,'(t)\right\|}$$

In this example, that means we are going to need the chain rule or the quotient rule to work this out. I'll use the chain rule since it means I won't have to find a common denominator later.

$$\vec{T}(t)=\left(10+4t^2\right)^{-1/2}\left(3\vec{i}-\vec{j}+2t\vec{k}\right)$$

$$\vec{T}'(t)=-\frac{1}{2}\left(10+4t^2\right)^{-3/2}(8t)\left(3\vec{i}-\vec{j}+2t\vec{k}\right)+\left(10+4t^2\right)^{-1/2}\left(2\vec{k}\right)=\frac{-4t\left(3\vec{i}-\vec{j}+2t\vec{k}\right)}{\left(10+4t^2\right)^{3/2}}+\frac{\left(10+4t^2\right)\left(2\vec{k}\right)}{\left(10+4t^2\right)^{3/2}}$$

$$=\frac{\left(-12t\vec{i}+4t\vec{j}-8t^2\vec{k}\right)+\left(20+8t^2\right)\vec{k}}{\left(10+4t^2\right)^{3/2}}=\frac{-12t\vec{i}+4t\vec{j}+20\vec{k}}{\left(10+4t^2\right)^{3/2}}$$

We next need the magnitude of this vector.

$$\vec{T}'(t)=\frac{-12t\vec{i}+4t\vec{j}+20\vec{k}}{\left(10+4t^2\right)^{3/2}}$$

$$\left\|\vec{T}'(t)\right\|=\frac{\sqrt{144t^2+16t^2+400}}{\left(10+4t^2\right)^{3/2}}=\frac{\sqrt{160t^2+400}}{\left(10+4t^2\right)^{3/2}}=\frac{4\sqrt{10t^2+25}}{\left(10+4t^2\right)^{3/2}}$$

Multiply the former by the reciprocal of the latter to get the unit normal vector. You notice that some parts cancel out.

$$\vec{N}(t)=\frac{\vec{T}'(t)}{\left\|\vec{T}'(t)\right\|}=\frac{-12t\vec{i}+4t\vec{j}+20\vec{k}}{\left(10+4t^2\right)^{3/2}}\cdot\frac{\left(10+4t^2\right)^{3/2}}{4\sqrt{10t^2+25}}=\frac{-12t\vec{i}+4t\vec{j}+20\vec{k}}{4\sqrt{10t^2+25}}=\frac{-3t\vec{i}+t\vec{j}+5\vec{k}}{\sqrt{10t^2+25}}$$

You can check for yourself that the vector is a unit vector now. The magnitude of the numerator is the denominator. But you see how the normalizing interacted with the result. This would not have been the answer obtained from just normalizing the acceleration. We can only get away with that is the magnitude of the tangent vector is constant.

Now, lastly the binormal vector: at least we don't have to take any more derivatives. The binormal is a vector perpendicular to both and so is defined as $\vec{B}=\vec{T}\times\vec{N}$. We will use properties of the cross product to pull out the normalizing factors.

$$\vec{B}=\vec{T}\times\vec{N}=\left(10+4t^2\right)^{-1/2}\left(10t^2+25\right)^{-1/2}\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\3&-1&2t\\-3t&t&5\end{vmatrix}=\frac{\left(-5-2t^2\right)\vec{i}-\left(15+6t^2\right)\vec{j}+\left(3t-3t\right)\vec{k}}{\sqrt{\left(10+4t^2\right)\left(10t^2+25\right)}}$$

$$=\frac{-\left(2t^2+5\right)\vec{i}-\left(6t^2+15\right)\vec{j}}{\sqrt{40t^4+200t^2+250}}$$

A little bit of algebra will show that this vector is also a unit vector since the denominator is the magnitude of the numerator.

If we are given at point to analyze the function at, we must do this after the equation has been obtained. If you are given a point in rectangular coordinates rather than in terms of t, you will need to use the original parameterization to find the t value.

Remember, everything we have done in this example is to produce unit vectors. In the next examples we will be working with some vectors, but also some planes, and we will not need to use unit vectors. What happens next with surfaces is quite different from what happens with curves. Find a way to separate the two in your mind.

When we intersect two surfaces, we get a curve in 3-space. We can also use this technique to find the tangent vector to the curve of intersection, once we've parameterized the intersection.

Practice Problems. Find the unit tangent and unit normal vectors for the curves at the given point. If the magnitude of the tangent and normal vectors are constant, also find the binormal vector.

1. $\vec{r}(t) = 6\cos t\vec{i} + 2\sin t\vec{j}, t = \frac{\pi}{3}$
2. $\vec{r}(t) = 2\sin t\vec{i} + 2\cos t\vec{j} + 4\vec{k}, P\left(\sqrt{2}, \sqrt{2}, 4\right)$
3. $\vec{r}(t) = \ln t\vec{i} + (t+1)\vec{j}, t = 2$
4. $\vec{r}(t) = \left(t^3 - 4t\right)\vec{i} + 2t^2\vec{j}, t = 1$
5. $\vec{r}(t) = 4t\vec{i} - 4t\vec{j} + 2t\vec{k}, t = 2$

Tangent Planes to a Surface, and Normal Lines in 3-space.
We need to recall how to write the equation of a plane in 3-space. To do this we need a vector normal to the plane, say, <a,b,c>, and a point in the plane, say (h,k,l). The equation of the plane then is: $a(x-h)+b(y-k)+c(z-l)=0$.

The equation of line is done similarly, but with notable differences. Using the same vector and a point on the line, the symmetric form of the line is $\frac{x-h}{a} = \frac{y-k}{b} = \frac{z-l}{c}$.

To do this for a surface, we will replace the vector <a,b,c> with the gradient of the equation of the surface, $gradF = \nabla F(x,y,z)$, evaluated at the point.

Example 2. Find the tangent plane to the graph $x^2 + 3y + z^3 = 9$ at the point (2,-1,2). Find the normal line at the same point.

First, we need to find the master F function, by putting all the variables and constants on one side of the equation.

$$F(x,y,z) = x^2 + 3y + z^3 - 9$$

Next we need some partial derivatives. And then evaluate it at the given point.

$$\nabla F(x,y,z) = \frac{\partial F}{\partial x}\vec{i} + \frac{\partial F}{\partial y}\vec{j} + \frac{\partial F}{\partial z}\vec{k} = 2x\vec{i} + 3\vec{j} + 3z^2\vec{k}$$
$$\nabla F(2,-1,2) = 4\vec{i} + 3\vec{j} + 12\vec{k}$$

Now combine this vector with our point to form the tangent plane.

$$4(x-2) + 3(y+1) + 12(z-2) = 0$$

You can distribute and collect the constants if you like, or solve for z, but that's not necessary unless you want to graph it.

Lastly, we need the normal line. Using the symmetric form of a line we get:

$$\frac{x-2}{4} = \frac{y+1}{3} = \frac{z-2}{12}$$

We can look at the graph and tangent plane together.

Practice Problems. Find the equations of the tangent plane and the normal line to the curve at the given point.

6. $x^2 + 4y^2 + z^2 = 36, P(2,-2,4)$
7. $x^2 + y^2 + z = 9, P(1,2,4)$
8. $y \ln xz^2 = 2, P(e,2,1)$
9. $xyz = 10, P(1,2,5)$
10. $2xy - z^3 = 0, P(2,2,2)$
11. What would be the condition needed to make the tangent plane to the graph $z = 3x^2 + 2y^2 - 3x + 4y - 5$ horizontal? Find that point. [Hint: when a plane is horizontal, what is the normal to the plane?]

Tangent Planes to a Surface, and Normal Lines to Parametric Surfaces in 3-space.
Parameterized surfaces in 3-space have 2 parametric variables, so we need some other method for finding tangent planes and normal vectors. If our parametric variables are u and v, we find the normal vector to the surface by the formula $\vec{r_u} \times \vec{r_v}$ where these component vectors are the partial derivatives of our parameterized surface, instead of using the gradient.

Example 3. Find the tangent plane to the graph $\vec{r}(u,v) = 2u\cos v\vec{i} + 3u\sin v\vec{j} + u^2\vec{k}$ at the point (0,6,4), and then find the equation of the normal line at the same point.

Start by taking the partial derivatives with respect to u and v of our surface.

$$\vec{r_u}(u,v) = 2\cos v\vec{i} + 3\sin v\vec{j} + 2u\vec{k}$$
$$\vec{r_v}(u,v) = -2u\sin v\vec{i} + 3u\cos v\vec{j} + 0\vec{k}$$

Next we need the cross product of these two vectors.

$$\vec{r_u}(u,v)\times\vec{r_v}(u,v) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2\cos v & 3\sin v & 2u \\ -2u\sin v & 3u\cos v & 0 \end{vmatrix}$$
$$= \left(0-6u^2\cos v\right)\vec{i} - \left(0+4u^2\sin v\right)\vec{j} + \left(6u\cos^2 v + 6u\sin^2 v\right)\vec{k}$$
$$= -6u^2\cos v\vec{i} - 4u^2\sin v\vec{j} + 6u\vec{k}$$

In order to get the vector at the point, we need to find out what u and v are at the point (0,6,4). We do this using the original parameterization.

$$x = 0 = 2u\cos v$$
$$y = 6 = 3u\sin v \Rightarrow \sin v = 1 \Rightarrow v = \frac{\pi}{2}$$
$$z = 4 = u^2 \Rightarrow u = 2$$

Using the data for z, we get two possible values. I am only considering one of them. Combining this with the information for y, we find a value for v. We get the same information from x, since u is non-zero, cos$v$=0. We can get a second parameterized point at $\left(-2,\dfrac{3\pi}{2}\right)$, but we just need one to obtain a vector.

Our normal vector is then $-6(2)^2\cos\frac{\pi}{2}\vec{i} - 4(2)^2\sin\frac{\pi}{2}\vec{j} + 6(2)\vec{k} = 0\vec{i} - 16\vec{j} + 12\vec{k}$. Using this information together with the point (0,6,4), we can obtain both the tangent plane equation and the normal line. (I use parametric form here, since we can't divide by zero in the symmetric form.)

$$0(x-0) - 16(y-6) + 12(z-4) = 0 \Rightarrow 12(z-4) = 16(y-6)$$
$$\vec{r}(t) = 0\vec{i} + (6-16t)\vec{j} + (4+12t)\vec{k}$$

Practice Problems. Find the tangent plane and normal line to the graph at the given point.

12. $\vec{r}(u,v) = u\vec{i} + v\vec{j} + uv\vec{k}, P(1,1,1)$
13. $\vec{r}(u,v) = 2\cos u\vec{i} + v\vec{j} + 2\sin u\vec{k}, P(2,4,0)$
14. $\vec{r}(u,v) = 3\cos v\cos u\vec{i} + 3\cos v\sin u\vec{j} + 5\sin v\vec{k}, P(3,0,0)$
15. $\vec{r}(u,v) = u\vec{i} + \frac{1}{4}v^3\vec{j} + v\vec{k}, P(-1,2,2)$