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Limits in 2 or more variables

Before beginning limits in multiple variables, it's a good idea to review limits in one variable. Recall that $\lim_{x\to c} f(x) = L$ is defined by saying that if $|x-c|<\delta$ this implies that $|f(x)-L|<\varepsilon$. In words, this says that if the difference between x and c is small enough (δ) then the value of f(x) will be within a small amount (ε) of L. In evaluating limits, we rarely employ this definition in practice, of course. Instead, we use algebraic properties derived from this definition, or equivalent expressions, to evaluate limits. When our function is already continuous, the value f(c) is always the value of L. But if the function is not continuous, maybe we get 0/0 meaning the function is not defined at that point, we use algebraic manipulations to simplify the problem so that the expression is defined without the zero in the denominator.

Among the techniques we used were:

- ✓ Factoring $\lim_{x\to 2}\frac{x^2+5x-14}{x^2+x-6} = \lim_{x\to 2}\frac{(x+7)(x-2)}{(x-2)(x+3)} = \lim_{x\to 2}\frac{x+7}{x+3} = \frac{9}{5}$
- ✓ Rationalizing denominators

$$\lim_{x\to 1^+}\frac{x-1}{\sqrt{x-1}} = \lim_{x\to 1^+}\frac{(x-1)\sqrt{x-1}}{\sqrt{x-1}\sqrt{x-1}} = \lim_{x\to 1^+}\frac{(x-1)\sqrt{x-1}}{x-1} = \lim_{x\to 1+}\sqrt{x-1} = 0$$

- ✓ Special limits $\lim_{x\to 0}\frac{\sin x}{x} = 1,\quad \lim_{x\to 0}\frac{1-\cos x}{x} = 0,\quad \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$
- ✓ Applying log rules

$$\lim_{x\to 0} x^x = L \Rightarrow \lim_{x\to 0}\ln\left(x^x\right) = \ln L \Rightarrow \lim_{x\to 0} x\ln x = \lim_{x\to 0}\frac{\ln x}{x^{-1}} = \lim_{x\to 0}\frac{x^{-1}}{-x^{-2}} = \lim_{x\to 0}[-x] = 0 \Rightarrow e^0 = 1 = L$$

- ✓ L'Hôpital's Rule (this was also used in the last example)

Many of these techniques can still be employed in multiple variables, but L'Hôpital's is for one variable only (if we want to use it, we will have to reduce the problem to a single variable first). You may wish to go back and review one variable limits more carefully because we will assume such techniques and employ them from time to time.

When we start working with multiple variables, our definition of a limit has to change a bit. Now $\lim_{(x,y)\to(c,d)} f(x,y) = L$ in the two variable case, is defined by: if $\sqrt{(x-c)^2+(y-d)^2} < \delta$ this implies that $|f(x,y)-L|<\varepsilon$. We can use any metric for the distance between (x,y) and (c,d), but the standard Euclidean distance formula is typical. As before, this just means that if (x,y) gets close enough (δ) to (c,d), then the difference between f(x,y) and L will be as small as we like (ε). This can be readily expanded to any number of variables, for instance, $\lim_{(x,y,z)\to(c,d,e)} f(x,y,z) = L$ is defined as if $\sqrt{(x-c)^2+(y-d)^2+(z-e)^2} < \delta$ this implies that $|f(x,y,z)-L| < \varepsilon$. And so forth. Such a definition exists in all coordinate systems.

However, proving that a limit exists in multiple variables is considerably more complicated than proving it in just one variable. In the one-variable case, we just need to worry about approaching the limit from the right side, and approaching from the left side. These are easy to each exhaustively, even if you have to do it numerically. When working with more than one variable, there are an infinite number of potential paths to a limit. So, very often, we are reduced to proving that a limit **does not exist** rather than proving that it does. We just need to find one path to the limit that is different than at least one other path. Finding that special path can be tricky, but there are a few things we can look for. Another strategy is to switch coordinate systems. Our goal will be to reduce the problem to a single variable, and possibly to apply algebra techniques that we used in the past.

Let's consider some examples. Before doing anything, always consider first if the function is continuous. If it is, just plug in the values. If it's not, that's when things get tricky. Our examples will consider only the non-continuous examples, but don't overthink the continuous ones.

Example 1. $\lim_{(x,y)\to(0,2)} \dfrac{y\sin x}{x}$

One possible trick here is to try separating variables.

$$\lim_{(x,y)\to(0,2)} \frac{y\sin x}{x} = \lim_{x\to 0}\frac{\sin x}{x}\cdot \lim_{y\to 2} y = 1\cdot 2 = 2$$

This usually isn't possible, but when it is, take advantage of it.

Example 2. $\lim_{(x,y)\to(0,0)} \dfrac{x-y}{\sqrt{x-y}}$

Here, let x-y=u, and when (x,y)=(0,0), then u=0

$$\lim_{(x,y)\to(0,0)} \frac{x-y}{\sqrt{x-y}} = \lim_{u\to 0}\frac{u}{\sqrt{u}} = \lim_{u\to 0}\sqrt{u} = 0$$

Example 3. $\lim_{(x,y)\to(0,0)} \dfrac{x^2y^2}{3y^4+x^4}$

This problem won't reduce nicely, so we will try various "paths" to the origin to see if we can get one result different than the others. Failing to find such a path does not mean that the desired path doesn't exist. We may not have been creative enough. Be extremely wary of this on exams.

There are a number of standard paths to try.

a) Let x=0 (this path is along the y-axis)

$$\lim_{(0,y)\to(0,0)} \frac{0\cdot y^2}{3y^4+0^4} = \lim_{y\to 0}\frac{0}{3y^4} = 0$$

b) Let y=0 (this path is along the x-axis)

$$\lim_{(x,0)\to(0,0)} \frac{x^2\cdot 0}{3\cdot 0^4+x^4} = \lim_{x\to 0}\frac{0}{x^4} = 0$$

c) Let y=kx (this is any linear path not along the axes, i.e. k≠0)

$$\lim_{(x,kx)\to(0,0)} \frac{x^2(kx)^2}{3(kx)^4+x^4} = \lim_{x\to 0}\frac{k^2x^4}{3k^4x^4+x^4} = \lim_{x\to 0}\frac{k^2x^4}{x^4(3k^4+1)} = \frac{k^2}{3k^4+1}$$

This last example is not equal zero unless k=0 (but we explicitly excluded that case because we dealt with it in (b)). This means that this limit **does not exist**.

You must be very careful choosing paths, though. While we are frequently looking at limits at (x,y)=(0,0), it is not universally true. When evaluating paths, we need to choose paths that approach the point we are interested in. Choosing the path y=0, if we are approaching the point (2,1) will not tell us anything (and will be quite wrong!).

Example 4. $\lim_{(x,y)\to(3,0)} \frac{4x+\ln(1+xy)}{1+x+y}$

a) Let x=3

$$\lim_{(3,y)\to(3,0)} \frac{4\cdot 3+\ln(1+3y)}{1+3+y} = \lim_{y\to 0}\frac{12+\ln(1+3y)}{4+y} = \frac{12+\ln 1}{4} = \frac{12}{4} = 3$$

b) Let y=0

$$\lim_{(x,0)\to(3,0)} \frac{4x+\ln(1+x\cdot 0)}{1+x+0} = \lim_{x\to 3}\frac{4x+\ln 1}{1+x} = \frac{12}{4} = 3$$

c) Let $x=3e^y$. This seems like a crazy path at first, but when $y\to 0, x\to 3$

$$\lim_{(3e^y,y)\to(3,0)} \frac{4\cdot 3e^y+\ln(1+3e^y\cdot y)}{1+3e^y+y} = \lim_{y\to 0}\frac{12e^y+\ln(1+3ye^y)}{1+3e^y+y} = \frac{12+\ln 1}{1+3+0} = \frac{12}{4} = 3$$

This looked pretty horrible, but nothing exceptional happened once we plugged in 0.

d) Let x=y+3

$$\lim_{(y+3,y)\to(3,0)} \frac{4(y+3)+\ln(1+(y+3)y)}{1+y+3+y} = \lim_{y\to 0}\frac{4y+12+\ln(y^2+3y+1)}{2y+4} = \frac{12}{4} = 3$$

So, how can I claim that the limit here is really 3? Well, we have to consider the continuity of the function. The denominator is undefined only when $x+y=-1$, but that's not a condition satisfied by the point (3,0). What about the numerator? 4x is continuous everywhere, but ln(1+xy) is undefined when xy≤ -1. Again, that's not a condition satisfied by our point, so both the numerator and denominator are continuous everywhere around the point (3,0) and so the limit is indeed just 3.

Example 5. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2}$

Start this problem in the usual way.

a) Let x=0 (along the y-axis)

$$\lim_{(0,y)\to(0,0)} \frac{0^2y}{0^4+y^2} = \lim_{y\to 0}\frac{0}{y^2} = 0$$

b) Let y=0 (along the x-axis)

$$\lim_{(x,0)\to(0,0)} \frac{x^2 0}{x^4+0^2} = \lim_{x\to 0}\frac{0}{x^4} = 0$$

c) Let y=kx (along a straight line to the origin, k≠0)

$$\lim_{(x,kx)\to(0,0)} \frac{x^2kx}{x^4+(kx)^2} = \lim_{x\to 0}\frac{kx^3}{x^4+k^2x^2} = \lim_{x\to 0}\frac{kx^3}{x^2(x^2+k^2)} = \lim_{x\to 0}\frac{kx}{x^2+k^2} = \frac{0}{k^2} = 0$$

So far so good, but what other path can we try? We want a path that will simplify the denominator. Here we have x^4 and y^2 so if we let $y=x^2$, these terms will have the same power. Let's try that.

d) Let $y=kx^2$ (where k≠0)

$$\lim_{(x,kx^2)\to(0,0)} \frac{x^2kx^2}{x^4+(kx^2)^2} = \lim_{x\to 0}\frac{kx^4}{x^4+k^2x^4} = \lim_{x\to 0}\frac{kx^4}{x^4(1+k^2)} = \frac{k}{1+k^2}$$

This last result is never equal 0 unless k=0, so we know that the limit **does not exist**.

For these curved paths, we want the variable to cancel out of the denominator, and don't be afraid to use fractional powers if need be. Whatever will get the two to be equal. You can try the same trick with the numerator.

Practice Problems.

1. $\lim\limits_{(x,y,z)\to(2,1,-1)} 3x^2z + yx\cos(\pi x - \pi z)$
2. $\lim\limits_{(x,y)\to(3,1)} \dfrac{xy}{x+y}$
3. $\lim\limits_{(x,y)\to(1,1)} \dfrac{xy}{x^2+y^2}$

4. $\lim_{(x,y)\to(1,1)} \frac{xy-1}{1+xy}$

5. $\lim_{(x,y)\to(1,-1)} \frac{x^2 y}{1+xy^2}$

6. $\lim_{(x,y)\to(0,1)} \frac{\arccos\left(\frac{x}{y}\right)}{1+xy}$

7. $\lim_{(x,y)\to(\frac{\pi}{4},2)} y\cos xy$

8. $\lim_{(x,y)\to(0,0)} \frac{1}{x+y}$

9. $\lim_{(x,y,z)\to(-2,1,0)} xe^{yz}$

10. $\lim_{(x,y)\to(2,1)} \frac{x-y-1}{\sqrt{x-y}-1}$

11. $\lim_{(x,y)\to(0,0)} \frac{x^2}{(1+x^2)(1+y^2)}$

12. $\lim_{(x,y)\to(0,0)} \frac{x-y}{\sqrt{x}-\sqrt{y}}$

13. $\lim_{(x,y)\to(0,0)} x+4y+1$

14. $\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz+xz}{x^2+y^2+z^2}$

15. $\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz^2+xz^2}{x^2+y^2+z^2}$

16. $\lim_{(x,y)\to(0,0)} 1-\frac{\cos(x^2+y^2)}{x^2+y^2}$

17. $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

18. $\lim_{(x,y)\to(0,0)} \frac{2x-y^2}{2x^2+y}$

19. $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{\sqrt{x^2+y^2}}$

20. $\lim_{(x,y)\to(0,0)} \frac{\sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}$

21. $\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$

22. $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{xy}$

23. $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^3+y^2}$

24. $\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2}$

25. $\lim_{(x,y,z)\to(0,0,0)} \arctan\left[\frac{1}{x^2+y^2+z^2}\right]$

26. $\lim_{(x,y)\to(0,1)} \arctan\left[\frac{x^2+1}{x^2+(y-1)^2}\right]$

27. $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+3y^4}$

28. $\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+y^2}$

29. $\lim_{(x,y)\to(0,0)} \frac{4x^2+y^2}{x^4+3y^4}$

30. $\lim_{(x,y)\to(0,0)} \frac{xy}{3+x^2y^2}$

31. $\lim_{(x,y)\to(0,0)} \frac{x+2y^2}{3x^2+2y^2}$

32. $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{3x^2+2y^2}}$

33. $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+3y^2}$

34. $\lim_{(x,y)\to(0,0)} \frac{-3x^3-y^2}{3x^3+2y^2}$

35. $\lim_{(x,y)\to(0,0)} \frac{x^2y^5}{2x^4+3y^{10}}$

36. $\lim_{(x,y)\to(0,0)} \frac{2x^3y^2}{x^6+y^4}$

37. $\lim_{(x,y)\to(0,0)} \frac{xy^2}{3x^2+2y^2}$

38. $\lim_{(x,y)\to(0,0)} \frac{3x^3+2\sqrt{y}}{x^2+y^2}$

39. $\lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{x^2+3y^2}$

40. $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{2x^2+3y^4}$

Another possible technique to consider is to convert to a different coordinate system. This technique may allow to find non-existence limits more easily. Not all problems can be solved this way, and the technique works best when the degree of terms in the denominator is the same, or if terms everywhere have the same degree, or if identities can be applied readily without a lot of algebra. Most importantly, *we should be approaching the origin*. While converting can we used in theory away from the origin, this method will become enormously more complicated.

Example 6. $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ (#17 from the practice problems)

Convert this problem to polar coordinates using $x = r\cos\theta, \quad y = r\sin\theta, \quad x^2+y^2=r^2$. In our limit, we allow r to go to zero, but leave θ alone. If our answer depends on θ, then the limit does not exist since different angles will give different values.

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{(r,\theta)\to(0,\theta)} \frac{r^2\cos\theta\sin\theta}{r^2} = \cos\theta\sin\theta$$

This limit **does not exist**.

You can also use this technique in three-variable problems.

Example 7. $\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2}$ (#24 from the practice problems)

If we convert only (x,y) to (r,θ), this is equivalent to using cylindrical coordinates. This will reduce a three-variable problem to the more familiar two-variable problem.

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2} = \lim_{(r,\theta,z)\to(0,\theta,0)} \frac{r^2z\cos\theta\sin\theta}{r^2+z^2}$$

You can treat θ like a constant for now. You can see fairly readily that if you approach the origin along $r \to 0$, the limit will be 0, and if you approach along the path $r = z$, you also get 0.

$$\lim_{(r,\theta,z)\to(0,\theta,z)} \frac{r^2 z\cos\theta\sin\theta}{r^2+z^2} = \lim_{r\to 0}\frac{0}{z^2} = 0$$

$$\lim_{(z,\theta,z)\to(0,\theta,0)} \frac{z^3\cos\theta\sin\theta}{2z^2} = \lim_{z\to 0}\frac{z\cos\theta\sin\theta}{2} = 0$$

It turns out that in this example, the degree of the numerator is one degree higher than the degree of the denominator, which is why if you reduce it to a single variable, L'Hôpital's Rule will preserve the variables in the numerator longer than in the denominator, and so the function will be 0 here.

But even more clearly in this problem, you can convert instead to spherical using $x=\rho\sin\varphi\cos\theta,\quad y=\rho\sin\varphi\sin\theta,\quad z=\rho\cos\varphi,\quad x^2+y^2+z^2=\rho^2$. And we will let ρ go to zero, and leave the other angles alone. If there is any angle information left at the end when we are finished, we will know the limit does not exist.

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2} = \lim_{(\rho,\varphi,\theta)\to(0,\varphi,\theta)} \frac{\rho^3\sin^2\varphi\cos\theta\sin\theta\cos\varphi}{\rho^2} = \lim_{\rho\to 0}\rho\sin^2\varphi\cos\theta\sin\theta\cos\varphi = 0$$

While this looks like a mess until we get to the end, the limit being truly equal to zero is much clearer than when we converted to cylindrical.

Three-variable problems are enormously difficult and tedious if we work them out in rectangular coordinates because of the possible paths we have to consider, and there is no guarantee that we will get a definitive result. Most of the three-variable problems we will deal with will be either continuous, or will be easily reducible in spherical coordinates (or more rarely, cylindrical coordinates).

Practice Problems.

41. Convert the listed problems into polar or spherical coordinates (for 2- or 3-variable problems respectively) and recalculate the limits. Do your results from before agree?
    a. $\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz+xz}{x^2+y^2+z^2}$ (#14, in spherical)
    b. $\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz^2+xz^2}{x^2+y^2+z^2}$ (#15, in spherical)
    c. $\lim_{(x,y)\to(0,0)} 1-\frac{\cos(x^2+y^2)}{x^2+y^2}$ (#16, in polar)
    d. $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{\sqrt{x^2+y^2}}$ (#19, in polar)
    e. $\lim_{(x,y)\to(0,0)} \frac{\sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}$ (#20, in polar)

f. $\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$ (#21, in polar)

g. $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{xy}$ (#22, in polar)

h. $\lim_{(x,y,z)\to(0,0,0)} \arctan\left[\frac{1}{x^2+y^2+z^2}\right]$ (#25, in spherical)

i. $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+3y^4}$ (#27, in polar)

j. $\lim_{(x,y)\to(0,0)} \frac{4x^2+y^2}{x^4+3y^4}$ (#29, in polar)

k. $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{3x^2+2y^2}}$ (#32, in polar)

l. $\lim_{(x,y)\to(0,0)} \frac{xy^2}{3x^2+2y^2}$ (#37, in polar)

m. $\lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{x^2+3y^2}$ (#39, in polar)

n. $\lim_{(x,y)\to(0,0)} \frac{3x^3+2\sqrt{y}}{x^2+y^2}$ (#38, in polar… tricky!!)