Lagrange Multipliers

Lagrange multipliers are a way of solving an optimization problem on a function subject to constraints. These kinds of problems are pretty common in the real world (modeling population subject to food supply, or modeling profit subject to labor costs, etc.). There is an unsophisticated method of solving these problems that often works, and we'll do an example of that at the end, but Lagrange multipliers are the technique we need to learn. It seems complicated at first because we have to introduce additional variables, but it tends to keep the algebra from getting too nasty otherwise. (The alternative method sometimes makes things easier, and sometimes the algebra just goes crazy.)

In problems where we want to use Lagrange multipliers we have a function *f* and a constraint function, we'll call *g*. These are functions of two or more variables. Problems can have more than one constraint, and we'll do an example of that later, but they can have only one function we are hoping to optimize.

Example 1. Minimize the function $f(x,y)=x^2+y^2$ subject to the constraint $x+2y=5$. (In this example, we will assume that *x* and *y* must both be greater than zero. This is common, but not universal.)

The first thing we need to do is rewrite the constraint function and set it equal to zero. This will then be *g*.

$$g(x,y)=x+2y-5$$

The next thing is to create the function we are going to optimize, which we will call *F*. We create this function by $F(x,y,\lambda)=f(x,y)-\lambda g(x,y)$. Here, that's

$$F(x,y,\lambda)=x^2+y^2-\lambda x-2\lambda y+5\lambda$$

I've distributed through the λ because I find it easier to keep the signs straight later.

F is now a function of three variables: *x*, *y*, *λ*, but it now incorporates the constraint. In order to optimize this equation, we need to find the critical point(s), so we need all three first partial derivatives.

$$F_x(x,y,\lambda)=2x-\lambda$$
$$F_y(x,y,\lambda)=2y-2\lambda$$
$$F_\lambda(x,y,\lambda)=-x-2y+5$$

When we set these equal to zero, we get the following:

$$2x-\lambda=0\Rightarrow 2x=\lambda$$
$$2y-2\lambda=0\Rightarrow y=\lambda$$
$$-x-2y+5=0\Rightarrow x+2y=5$$

You'll notice that the λ-derivative is just the original constraint. I've solved the other two equations for λ, and we can use that to reduce the last equation to one variable and solve the system easily.

$$2x = \lambda, y = \lambda$$
$$x + 2y = 5 \Rightarrow \frac{1}{2}\lambda + 2\lambda = 5 \Rightarrow \frac{5}{2}\lambda = 5 \Rightarrow \lambda = 2$$
$$x = 1, y = 2$$

So, our critical point is at (1,2). We don't need the λ any longer since it's part of the technique and not the solution. We can use our critical point to solve the optimization problem by putting it back into the original equation.

$$f(1,2) = (1)^2 + (2)^2 = 5$$

Our problem told us in advance that this was a minimum. We could prove this by using our matrix of second partials (remember, we'd need nine of them), but let's skip that.

Alternate solution method: Substitution.
This example is easy enough that I can show you the alternate solution method that does not involve the introduction of λ. However, even this will involve more algebra up front.

Minimize the function $f(x, y) = x^2 + y^2$ subject to the constraint $x + 2y = 5$.

First, solve the constraint function for one of the variables and replace it in the original equation.

$$x + 2y = 5 \Rightarrow x = 5 - 2y$$
$$f(x, y) = x^2 + y^2 \Rightarrow f(5-2y, y) = (5-2y)^2 + y^2$$

I've used the constraint to reduce the function by one variable (if I start with three, I'd end up with two if I have only one constraint). I will now use calculus of one variable to solve this. (You can use the D-test if you have two variables left.

$$f(5-2y, y) = (5-2y)^2 + y^2 = 25 - 20y + 5y^2$$
$$f'(5-2y, y) = -20 + 10y$$
$$-20 + 10y = 0 \Rightarrow 10y = 20 \Rightarrow y = 2$$
$$x = 5 - 2(2) = 1$$

I was able to get the same solution (1,2), and by taking a second derivative and seeing that its positive at this point, I can determine that it is indeed a minimum.

Be aware that you will be expected to use Lagrange multipliers, so think of this method as a means of checking your answer when the algebra isn't too messy, not a way to avoid learning the Lagrange multiplier method entirely.

Let's try a harder one.

Example 2. Maximize $f(x,y,z)=x^2+y^2+z^2$ subject to the constraint $x+y+z=1$.

Since $g(x,y,z)=x+y+z-1$, our *F* function is:

$$F(x,y,z,\lambda)=f(x,y,z)-\lambda g(x,y,z)=x^2+y^2+z^2-\lambda x-\lambda y-\lambda z+\lambda$$

Take our four first partial derivatives.

$$F_x(x,y,z,\lambda)=2x-\lambda$$
$$F_y(x,y,z,\lambda)=2y-\lambda$$
$$F_z(x,y,z,\lambda)=2z-\lambda$$
$$F_\lambda(x,y,z,\lambda)=-x-y-z+1$$

The last equation, when we set it equal to zero, is again our original constraint.

$$F_x(x,y,z,\lambda)=2x-\lambda=0\Rightarrow 2x=\lambda\Rightarrow x=\frac{1}{2}\lambda$$
$$F_y(x,y,z,\lambda)=2y-\lambda=0\Rightarrow 2y=\lambda\Rightarrow y=\frac{1}{2}\lambda$$
$$F_z(x,y,z,\lambda)=2z-\lambda=0\Rightarrow 2z=\lambda\Rightarrow x=\frac{1}{2}\lambda$$
$$F_\lambda(x,y,z,\lambda)=-x-y-z+1=0\Rightarrow x+y+z=1\Rightarrow \frac{3}{2}\lambda=1\Rightarrow \lambda=\frac{2}{3}$$
$$(x,y,z)=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$$

We were told in the problem set up that this would be a maximum, so there is no need to test further.

As a side note, using the alternative method outline above would involve squaring a trinomial, and this is not even that complicated an equation. What if there were two constraints, and more terms with several variables?

Let's consider one example where there are two constraints.

Example 3. Maximize $f(x,y,z)=xyz$ subject to the constraints $x+y+z=32, \mathit{and}\ x-y+z=0$.

We need to form our *F* function, but since we have two constraints, we'll need to solve both equations for zero and have a *g* and an *h* function. And we'll also need to introduce a second dummy variable. Our F function is now:

$$g(x,y,z) = x + y + z - 32$$
$$h(x,y,z) = x - y + z$$
$$F(x,y,z,\lambda,\mu) = f(x,y,z) - \lambda g(x,y,z) - \mu h(x,y,z) = xyz - \lambda x - \lambda y - \lambda z + 32\lambda - \mu x + \mu y - \mu z$$

And now, we'll need five first partial derivatives.

$$F_x(x,y,z,\lambda,\mu) = yz - \lambda - \mu$$
$$F_y(x,y,z,\lambda,\mu) = xz - \lambda + \mu$$
$$F_z(x,y,z,\lambda,\mu) = xy - \lambda - \mu$$
$$F_\lambda(x,y,z,\lambda,\mu) = -x - y - z + 32$$
$$F_\mu(x,y,z,\lambda,\mu) = -x + y - z$$

Setting each of these equal zero, we can go about solving them in any way we like, whatever way you can eliminate variables. Sometimes you will see things that jump out at you, or trying solving for the dummy variables.

$$\begin{array}{ll} yz - \lambda - \mu = 0 & \qquad xy - \lambda - \mu = 0 \\ \underline{xz - \lambda + \mu = 0} & \qquad \underline{xz - \lambda + \mu = 0} \\ yz + xz - 2\lambda = 0 & \qquad xy + xz - 2\lambda = 0 \end{array}$$

$$yz + xz = 2\lambda$$
$$xy + xz = 2\lambda \Rightarrow yz + xz = xy + xz \Rightarrow yz = xy \Rightarrow yz - xy = 0 \Rightarrow y(z - x) = 0$$

$$\begin{array}{ll} -x - y - z + 32 = 0 \Rightarrow & x + y + z = 32 \\ -x + y - z = 0 \qquad \Rightarrow & \underline{x - y + z = 0} \\ & 2x + 2x = 32 \Rightarrow x + z = 16 \end{array}$$

Using the first three derivatives, I add them to eliminate μ, and then setting both equal to 2λ, I eliminated λ as well. In the middle section I factored, and found that either *y*=0, or *x*=*z*. Using the constraints, I added and eliminated *y*, leaving me just an equation in *x* and *z*. I can now solve for one of the variables.

$$x + x = 16 \Rightarrow 2x = 16 \Rightarrow x = 8$$
$$z = 8$$
$$8 + y + 8 = 32 \Rightarrow 16 + y = 32 \Rightarrow y = 16$$

Once you find *x* and *z*, you can use any of the other equations to get *y*, but I used of the constraints. Our final point then is (8,16,8).

What about the possible solution y=0 you may be asking? It's true that if you are not told explicitly that the variables must be strictly greater than zero that this is a possible solution, you must go ahead and get more than one point.

$$y = 0 \Rightarrow x + z = 0 \Rightarrow x = -z$$
$$x + y + z = 32 \Rightarrow -z + y + z = 32 \Rightarrow y = 32$$

But this is a contradiction, so the y=0 solution isn't actually a solution to the system. But you should always check all possibilities.

Can you do this kind of problem by substitution? Sure. You can reduce it to two variables, subject to one constraint, or with some more algebra, reduce the constraints, and if you are lucky, reduce the equation down to one or two variables and no constraints. But, even if you can, you haven't really saved yourself the algebra. You've just done it up front. And taking derivatives isn't that hard is it?

Practice Problems:

1. Maximize $w=x^2-y^2$, subject to $x+2y-5=0$
2. Maximize $w=x^2-10x+y^2-14y+28$, subject to $x+y=10$
3. Maximize $w=xy+yz+xz$, subject to $x+y+z=1$, and $x-2y+3z=15$
4. Minimize $w=x^2+y^2+z^2$, subject to $x+2z=6$, and $x+y=12$