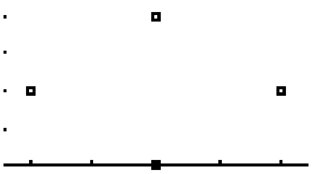
Jacobians and Change of Variable

When we worked with single variables, complicated functions could be simplified for integration with a change of variables. The same can be done with multiple variables, but the procedure is a bit more complicated. Change of variables in 2 or 3 dimensions can also be used to simplify a region of integration.

Example 1. Find the volume of the region lying below the surface $f(x, y) = (x + y)e^{x-y}$ over the region bounded by the parallelogram whose vertices are (4,0), (6,2), (4,4), (2,2).

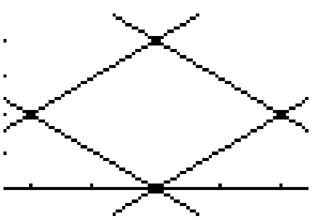
Step 1. For a problem like this, the first thing that needs to be done is graph the points and find the equations of the lines that connect them.



We need to calculate the slope of the lines between (2,2) and (4,4); (4,0) and (6,2); (4,0) and (2,2); and (4,4) and (6,2). Because we are told the region is a parallelogram, we expect only two slopes out of these lines since they have to be like in pairs.

$$m_1 = \frac{4-2}{4-2} = 1,\ m_2 = \frac{2-0}{6-4} = 1, m_3 = \frac{2-0}{2-4} = -1, m_4 = \frac{2-4}{6-4} = -1$$

Then we need to calculate the equations of each of these lines. And then put each one in standard form.

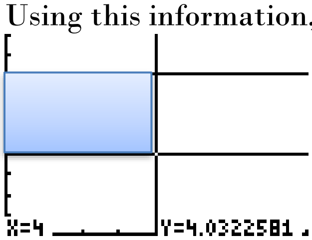


We can verify that the lines go through the points by adding them to our graph.
Line 1: $y - 2 = 1(x - 2) \rightarrow y = x \rightarrow x - y = 0$
Line 2: $y - 0 = 1(x - 4) \rightarrow y = x - 4 \rightarrow x - y = 4$
Line 3: $y - 0 = -1(x - 4) \rightarrow y = -x + 4 \rightarrow x + y = 4$
Line 4: $y - 2 = -1(x - 6) \rightarrow y = -x + 8 \rightarrow x + y = 8$

If we tried to integrate over this region as is, we'd have to divide the region into two pieces, and given the function we are integrating, it'd be hard work.

Step 2. Notice that the side of one pair of equations is x-y, and the other is x+y. These are going to be where we make our substitutions. Let u=x-y and v=x+y.

Using this information, we now have that u runs between 0 and 4, and v runs between 4 and



8. These will be our limits of integration. We can graph the region using these limits. The boundaries of the region represent a nice rectangle, which is easy to integrate over. The u and v can also be substituted back into our function to get a function of u and v instead of x and y.

$$f(u,v) = ve^u$$

Our integral is then $\int_0^4 \int_4^8 ve^u dA$.

Step 3. Before we can do any integrating, we have to find out how our change of variable affected the size of our area that we are integrating over. It may be merely scaled, or some change of variable problems will leave behind an additional variable component. To calculate this, we use the Jacobian:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Calculating this requires that we solve the pair of equations u=x-y and v=x+y for x in terms of u and v, and y in terms of u and v.

Adding them will eliminate the y's: $\begin{matrix} u=x-y \\ v=x+y \end{matrix} \rightarrow u+v=2x \rightarrow \frac{1}{2}(u+v)=x.$

Subtracting them will eliminate the x's: $\begin{matrix} u=x-y \\ -v=-x-y \end{matrix} \rightarrow u-v=-2y \rightarrow \frac{1}{2}(v-u)=y$

Take the derivative of the x equation to get the top row, and the y equation to get the bottom row.

$$\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}\left[\frac{1}{2}(u+v)\right] = \frac{1}{2}$$
$$\frac{\partial x}{\partial v} = \frac{\partial}{\partial v}\left[\frac{1}{2}(u+v)\right] = \frac{1}{2}$$
$$\frac{\partial y}{\partial u} = \frac{\partial}{\partial u}\left[\frac{1}{2}(v-u)\right] = -\frac{1}{2}$$
$$\frac{\partial y}{\partial v} = \frac{\partial}{\partial v}\left[\frac{1}{2}(v-u)\right] = \frac{1}{2}$$

Insert these into the matrix and take the determinant: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc.$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) - \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{2}$$

dA in our integral then becomes: $\frac{\partial(x,y)}{\partial(u,v)} dudv = \frac{1}{2} dudv$

Step 4. Now we can integrate.

$$\int_0^4 \int_4^8 ve^u \frac{1}{2} dvdu$$

It no longer matters which order you integrate in, but do be sure that the right limits go into the correct variable.

Our final answer then is: $12(e^4 - 1) \approx 643.1778$ ....

The procedure is pretty long to do from scratch, but we have some standard change of variables that we use all the time, such as switching to polar coordinates. Ever wonder what dxdy=rdrdθ and not just drdθ? The extra r comes from calculating the Jacobian for the switch. But since we use it all the time, we don't have to derive it from scratch each time. This procedure seems long because ever problem does have to be done from scratch. Though, sometimes we will be given equations that bind the region rather than, in the example, we had to derive even the boundary equations from scratch.

Practice Problems.
For each function f(x,y) calculate the region below it on the region described. Do this by changing to a convenient pair of variables. Sketch the region before and after the switch.

1. $f(x,y) = 60xy$ on the region bounded by the vertices (0,1), (1,2), (2,1), (1,0). This is a square.
2. $f(x,y) = y(x-y)$ on the parallelogram bounded by (0,0), (3,3), (7,3), (4,0).
3. $f(x,y) = e^{-\frac{xy}{2}}$ on the region bounded by $y = 2x, y = \frac{4}{x}, y = \frac{1}{4}x, y = \frac{1}{x}$.
4. $f(x,y) = ysin(xy)$ on the region bounded by xy=1, y=4, xy=4, and y=1.
5. $f(x,y) = (3x+2y)^2\sqrt{2y-x}$ on the parallelogram with vertices (0,0), (-2,3), (2,5), and (4,2).
6. $f(x,y) = \frac{xy}{1+x^2y^2}$ on the region bounded by the graphs of xy=1, xy=4, x=1, x=4.

Find the Jacobian for the three-dimensional change of coordinates:

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

You can calculate a three-dimensional determinant much like you would a cross-product. Or, you can enter the matrix in your calculator and use the DET feature.

1. $x = u(1-v), y = uv(1-w), z = uvw$
2. $x = u - v + w, y = 2uv, z = u + v + w$
3. $x = rcos\theta, y = rsin\theta, z = z$