

## Implicit Differentiation

Back in Calc I, we learned how to do implicit differentiation with two variables. In our typical example we used the  $x$ -variable as our independent variable and the  $y$ -variable as our implicit function variable. We expand on this process in this course with equations with three and four variables, but let's review the two variable case.

**Example 1.** Find  $\frac{dy}{dx}$  for the implicit function  $x^3 - xy + y^2 = \cos y$ .

Remember the general rule is to treat the  $x$ -variable normally, the  $y$ -variables by themselves as chain rules (take the derivative normally and then multiply by  $\frac{dy}{dx}$ ), and any terms that have both  $x$  and  $y$  will require product or quotient rules since  $y$  is implicitly  $y(x)$ .

$$\begin{aligned}\frac{d}{dx}[x^3 - xy + y^2] &= \frac{d}{dx}[\cos y] \\ \frac{d}{dx}(x^3) - \frac{d}{dx}(xy) + \frac{d}{dx}(y^2) &= \frac{d}{dx}(\cos y) \\ 3x^2 - \left(x \frac{dy}{dx} + (1)y\right) + 2y \frac{dy}{dx} &= -\sin y \frac{dy}{dx} \\ 3x^2 - x \frac{dy}{dx} - y + 2y \frac{dy}{dx} &= -\sin y \frac{dy}{dx}\end{aligned}$$

Once you've taken the derivative of every term, collect the  $\frac{dy}{dx}$  terms on one side and the remaining terms on the other.

$$3x^2 - y = -\sin y \frac{dy}{dx} + x \frac{dy}{dx} - 2y \frac{dy}{dx}$$

Factor out the  $\frac{dy}{dx}$ , and divide by the parentheses to get the  $\frac{dy}{dx}$  by itself.

$$\begin{aligned}3x^2 - y &= (-\sin y + x - 2y) \frac{dy}{dx} \\ \frac{3x^2 - y}{-\sin y + x - 2y} &= \frac{dy}{dx}\end{aligned}$$

When I was first learning this, I found the  $\frac{dy}{dx}$  confusing (too much notation), so you may find it helpful to use the less visually busy  $y'$  notation instead.

$$3x^2 - xy' - y + 2yy' = -\sin yy' \quad (\text{this is the fourth line from above})$$

In the case of several variables, things get a little more complicated because we may have to do this more than once, and some variables will be held constant. So we are treating variables three different ways.

**Example 2.** Find  $\frac{\partial z}{\partial x}$  for the implicit function  $3x^2 - xy + xz - 2yz^3 + z^4 = \frac{z}{x}$ .

Look at the partial derivative we are being asked for. It tells us that the  $x$ -variable is the independent variable we are interested in here, so this variable gets treated normally. The  $z$ -variable is the function variable, so it will get treated with the chain rule as we did with  $y$  in the previous example. The  $y$ -variable is not in the notation, so it's being treated like a constant for the partial derivative. Products or quotients of  $x$  and  $z$  need the product or quotient rules, but other combinations involve a variable we are treating like a constant, so they are treated like we would the remaining variable. Here, I will use the  $z_x$  notation instead of the  $\frac{\partial z}{\partial x}$  once I actually take derivatives.

$$\begin{aligned}\frac{\partial}{\partial x}(3x^2 - xy + xz - 2yz^3 + z^4) &= \frac{\partial}{\partial x}\left(\frac{z}{x}\right) \\ \frac{\partial}{\partial x}(3x^2) - \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial x}(xz) - \frac{\partial}{\partial x}(2yz^3) + \frac{\partial}{\partial x}(z^4) &= \frac{\partial}{\partial x}\left(\frac{z}{x}\right) \\ 6x - y + (xz_x + 1z) - 6yz^2z_x + 4z^3z_x &= \frac{xz_x - 1z}{x^2}\end{aligned}$$

As before, collect terms on one side that have the  $z_x$  factor, and then divide out the coefficients to isolate.

$$\begin{aligned}6x - y + xz_x + z - 6yz^2z_x + 4z^3z_x &= \frac{xz_x}{x^2} - \frac{z}{x^2} \\ 6x - y + \frac{z}{x^2} + z &= \frac{z_x}{x} - xz_x + 6yz^2z_x - 4z^3z_x \\ 6x - y + \frac{z}{x^2} + z &= \left(\frac{1}{x} - x + 6yz^2 - 4z^3\right)z_x \\ \frac{6x - y + \frac{z}{x^2} + z}{\frac{1}{x} - x + 6yz^2 - 4z^3} &= z_x = \frac{\partial z}{\partial x}\end{aligned}$$

In a problem like this, though, don't leave complex fractions.

$$\frac{6x^3 - yx^2 + z + zx^2}{x - x^3 + 6x^2yz^2 - 4x^2z^3} = z_x = \frac{\partial z}{\partial x}$$

Sometimes you will be asked to find all partial derivatives of a function implicitly. There is a hierarchy of function variables to keep in mind when doing these problems.

- If there are two variables,  $y$  is the function variable; so find  $\frac{dy}{dx}$ , but NOT  $\frac{dx}{dy}$ .

- If there are three variables, then  $z$  is the function variable; so find  $\frac{\partial z}{\partial x}$  and  $\frac{\partial z}{\partial y}$ , but NOT  $\frac{\partial x}{\partial z}$ ,  $\frac{\partial x}{\partial y}$ ,  $\frac{\partial y}{\partial x}$  or  $\frac{\partial y}{\partial z}$ .
- If there are four variables, then  $w$  is the function variable and you should find  $\frac{\partial w}{\partial x}$ ,  $\frac{\partial w}{\partial y}$ ,  $\frac{\partial w}{\partial z}$ , but NOT any of the other possible combinations (there are a lot of possibilities, so just don't).
- In more uncommon circumstances, ask if you aren't sure, but the key is to determine which are the function (dependent) variables (on top) and which are the independent (or sometimes parametric) variables (on bottom). The latter are typically  $t, s, u, v$ . In polar coordinates,  $\vartheta$  is independent and  $r$  is the function variable. In cylindrical coordinates,  $z$  is the function variable and  $r$  and  $\vartheta$  are both independent. In spherical,  $\rho$  is the function variable and  $\vartheta$  and  $\varphi$  are the independent variables.

**Practice Problems.** Find the implicit derivative, or first partial derivatives of the following implicit functions. Don't find more than you need.

1.  $3x^2 - xy + xz - 2yz^3 + z^4 = \frac{z}{x}$  (we found one in example 2; find the other)

2.  $4x^3 - \sin(xy) = ye^x$

3.  $x^5 - xyz + z^4 = \ln\left(\frac{x}{y}\right)$

4.  $x^2 - 2y^2z^3 + z = \frac{\sqrt{1 + xe^{y^2+z^2}}}{xyz}$

5.  $2x^2w - xyz - 2w \tan(zw) + e^{4w} = \frac{zy}{w}$

### Finding implicit derivatives with partial derivatives.

There is a second way to find implicit derivatives that is a bit easier (**you will still be expected to be able to do it both ways, so don't get too excited**). This method involves taking partial derivatives of a master function, as though all the variables were independent variables of some extra function variable, and then combining the partials to get the implicit derivative. The "master function" is formed by putting all the terms in our implicit function on one side and setting it equal to 0.

**Example 3.** Consider our equation from Example 1:  $x^3 - xy + y^2 = \cos y$ . Let's call our master function

$F(x, y)$ . Move the  $\cos(y)$  to the left and we will have it:  $F(x, y) = x^3 - xy + y^2 - \cos y$ . We need  $\frac{\partial F}{\partial x}$

and  $\frac{\partial F}{\partial y}$ . Remember here we are doing partial derivatives normally, so when we take the derivative for

$x$ , all other variables are treated like constants.

$$\frac{\partial F(x, y)}{\partial x} = \frac{\partial}{\partial x}(x^3 - xy + y^2 - \cos y) = 3x^2 - y$$

$$\frac{\partial F(x, y)}{\partial y} = \frac{\partial}{\partial y}(x^3 - xy + y^2 - \cos y) = -x + 2y + \sin y$$

Then we can find the implicit derivative through the formula:  $\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$

Note some important features of this formula: 1) the negative sign, and 2) the order we are dividing the partials in is the opposite of the derivative notation.

$$\frac{dy}{dx} = -\frac{3x^2 - y}{-x + 2y + \sin y} = \frac{3x^2 - y}{x - 2y - \sin y} \text{ if we distribute the negative through the denominator.}$$

Notice that we have the same solution as we had in Example 1, give or take the order of terms.

For three and four variables the formulas are:

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{F_x}{F_z}, \quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{F_y}{F_z}$$

$$\frac{\partial w}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial w}} = -\frac{F_x}{F_w}, \quad \frac{\partial w}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial w}} = -\frac{F_y}{F_w}, \quad \frac{\partial w}{\partial z} = -\frac{\frac{\partial F}{\partial z}}{\frac{\partial F}{\partial w}} = -\frac{F_z}{F_w}$$

**Practice Problems.** Find the implicit derivatives using the formulas listed above. Verify that they agree with the results of the first set of practice problems (and Example 2).

6.  $3x^2 - xy + xz - 2yz^3 + z^4 = \frac{z}{x}$  (this is Example 2, do both derivatives)

7.  $4x^3 - \sin(xy) = ye^x$

8.  $x^5 - xyz + z^4 = \ln\left(\frac{x}{y}\right)$

9.  $x^2 - 2y^2z^3 + z = \frac{\sqrt{1 + xe^{y^2+z^2}}}{xyz}$

10.  $2x^2w - xyz - 2w \tan(zw) + e^{4w} = \frac{zy}{w}$