Relative Extrema in Two or More Variables

Relative Extrema.
Functions of one variable are easy. You take a derivative, set it equal zero to find the critical points. And the sign of the second derivative tells you whether the critical point is a maximum or a minimum. While, you're second derivative might be zero, and thus, you'll have to determine what's going on at the critical point some other way, this is basically all there is when it comes to finding relative extrema.

But when it comes to two or more variables, things get more complicated. We have more than one first partial derivative, and more than one second derivative. But we still do need all the first partials to be equal to zero. Let's look at an example with two variables.

Example 1. Find the critical points of $f(x, y) = x^2 - xy + 3x - 6y + 4y^2$.

The first step is to find the first partials.

$$f_x(x, y) = 2x - y + 3$$
$$f_y(x, y) = -x + 8y - 6$$

If both of these are set equal to zero, we have a system of equation, a linear system in this case.

$$\begin{cases} 2x - y + 3 = 0 \\ -x + 8y - 6 = 0 \end{cases} \rightarrow \begin{cases} 2x - y = -3 \\ -2x + 16y = 12 \end{cases} \rightarrow 15y = 9 \rightarrow y = \frac{3}{5} \rightarrow -x + 8\left(\frac{3}{5}\right) - 6 = 0 \rightarrow x = -\frac{6}{5}$$

Our critical point then is $\left(-\frac{6}{5}, \frac{3}{5}\right)$. Because the system is linear, there is only one critical point. (As long as the degree of the polynomial doesn't exceed degree two, a linear system is to be expected.)

To determine what kind of critical point this is: is it a minimum or a maximum we will need second derivatives. In equations more two or more variables, we could also have a saddle point (a point where the second derivatives of the two equations are going in opposite directions), and, of course, we could have a situation where we can't tell what is happening.

Find the all the second partials. This is a well-behaved function, so $f_{xy} = f_{yx}$ so we just need one of these.

$$f_{xx}(x, y) = 2$$
$$f_{xy}(x, y) = -1$$
$$f_{yy}(x, y) = 8$$

If the second derivatives depend on any variables, we will need to plug in the critical point and consider the second derivative test separately for each point.

The second partials test is also sometimes called the D-test. It looks like this for two variables:

$$f_{xx} f_{yy} - \left(f_{xy}\right)^2 = D$$

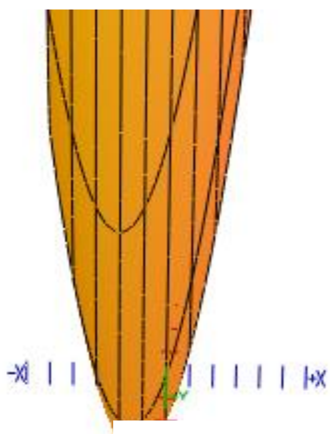
If D > 0 then the critical point is either a maximum or a minimum. If D < 0 then the critical point is a saddle point. If D = 0, then the second partials test fails, and we will have to use other means to determine what is happening: a graph, for instance. This latter case does not occur terribly often in textbook materials.

In this example, the second derivatives are already constants, so we can calculate D:

$$(2)(8)-(-1)^2=16-1=15$$

This is greater than zero, and so the critical point is either a maximum or a minimum. To determine which of these look at the sign of f_{xx} (or f_{yy}: they will give you the same info). If f_{xx} > 0 then the graph is concave up, and so the critical point is a minimum; if f_{xx} <0 then the graph is concave down, and so the critical point is a minimum, just like with equations of one variable.

The last step is to put the critical point back into the function to get the three-dimensional point where the minimum occurs: $\left(-\frac{6}{5},\frac{3}{5},-\frac{18}{5}\right)$.



That was a relatively easy example, so let's look at one with more than one critical point.

Example 2. Find the critical points of $f(x,y)=y^3-3yx^2-3y^2-3x^2+1$

Find the first partials.

$$f_x(x,y)=-6yx-6x$$
$$f_y(x,y)=3y^2-3x^2-6y$$

Solve the system by setting the equations equal to zero. Remember that when solving, especially in these non-linear systems, factor out variables, don't divide them out. If the variable could be zero, you will lose that information and not find all the critical points.

$$\begin{cases}-6yx-6x=0\\3y^2-3x^2-6y=0\end{cases}\Rightarrow\begin{cases}yx+x=0\\y^2-2y-x^2=0\end{cases}\Rightarrow x(y+1)=0$$

So far, we have x=0 or y= -1. Let's try these in the second equation.

$$y^2 - 2y - (0)^2 = 0 \Rightarrow y(y-2) = 0$$

If x=0, then y=0 or y= -2. Two possible points here: (0,0), and (0,-2). Let's try the other one possible value from the first equation.

$$(-1)^2 - 2(-1) - x^2 = 0 \Rightarrow x^2 = 1+2 \Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3}$$

We have two more points from this: $\left(\sqrt{3},-1\right),\left(-\sqrt{3},-1\right)$.

So now we need the second partials.

$$f_{xx}(x,y) = -6y-6$$
$$f_{xy}(x,y) = -6x$$
$$f_{yy}(x,y) = 6y-6$$

We need values for all four critical points to calculate D.

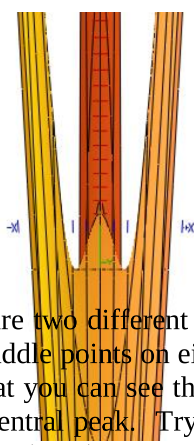
$$f_{xx}(0,0) = -6(0)-6 = -6$$
$$f_{xy}(0,0) = -6(0) = 0$$
$$f_{yy}(0,0) = 6(0)-6 = -6$$

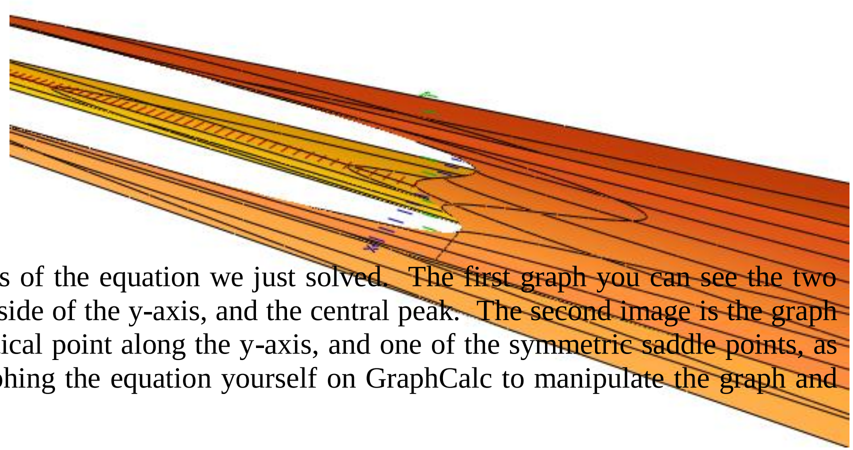
$$f_{xx}(0,-2) = -6(-2)-6 = 12-6 = 6$$
$$f_{xy}(0,-2) = -6(0) = 0$$
$$f_{yy}(0,-2) = 6(-2)-6 = -12-6 = -18$$

$$f_{xx}(\sqrt{3},-1) = -6(-1)-6 = 6-6 = 0$$
$$f_{xy}(\sqrt{3},-1) = -6(\sqrt{3}) = -6\sqrt{3}$$
$$f_{yy}(\sqrt{3},-1) = 6(-1)-6 = -6-6 = -12$$

$$f_{xx}(-\sqrt{3},-1) = -6(-1)-6 = 6-6 = 0$$
$$f_{xy}(-\sqrt{3},-1) = -6(-\sqrt{3}) = 6\sqrt{3}$$
$$f_{yy}(-\sqrt{3},-1) = 6(-1)-6 = -6-6 = -12$$

For (0,0), D= (-6)(-6)-0 = 36. This is >0, so it's a maximum or a minimum. Since $f_{xx} < 0$, this is a maximum. For (0,-2), D= (6)(-18)-0 = -108. This is < 0, so this is a saddle point. For the other two points, both give the same D value: -108. These are also saddle points.





The graphs are two different views of the equation we just solved. The first graph you can see the two symmetric saddle points on either side of the y-axis, and the central peak. The second image is the graph rotated so that you can see the critical point along the y-axis, and one of the symmetric saddle points, as well as the central peak. Try graphing the equation yourself on GraphCalc to manipulate the graph and better see what is going on.

The D-test formula can also be written as a matrix. It's the determinant of the second partials matrix:

$$\det\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = f_{xx}f_{yy} - (f_{xy})^2 .$$

The matrix form is how we can generalize this to equations with more than two variables. Critical points are still found from all the first partials being set equal zero. The second partials test is used to determine if we have a max/min, a saddle point, or cannot be determined just as with two variables. If the determinant of our second partials matrix > 0, then we have a max/min; if the determinant < 0, then we have a saddle point; and if the determinant = 0, then the test fails.

For three variables, the matrix looks like this:

$$\det\begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix} = f_{xx}\left(f_{yy}f_{zz} - f_{yz}f_{zy}\right) - f_{xy}\left(f_{yx}f_{fzz} - f_{yz}f_{zx}\right) + f_{xz}\left(f_{yx}f_{zy} - f_{yy}f_{zx}\right)$$

Keep in mind that the mixed second partials are not likely to be the same, so you will need to be careful of the order you are differentiating in, and you will need all nine of them.

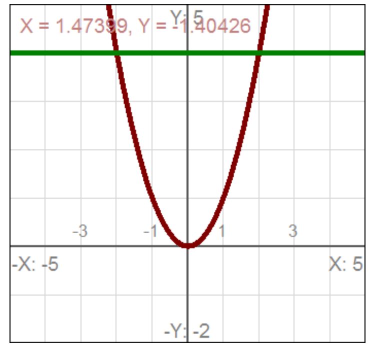
Practice Problems. Find all the critical points of each function, and use the second partials test to determine whether each critical point is a maximum, a minimum, a saddle point, or if the test fails. For #8, this equation is a hyperparaboloid (i.e. a 4-dimensional paraboloid); you can check the solution even if you can't graph it, by finding the vertex (generalize from the 3-dimensional case). However, trying doing it by the D-test.

1. $f(x, y) = xy$
2. $f(x, y) = 2x^2 + 2xy + y^2 + 2x - 3$
3. $f(x, y) = \sqrt{x^2 + y^2}$
4. $f(x, y) = \left(x^2 + 4y^2\right)e^{1-x^2-y^2}$
5. $f(x, y) = x^3 + y^3$
6. $f(x, y) = x^{2/3} + y^{2/3}$
7. $f(x, y) = x^3 + y^3 - 6x^2 + 9y^2 + 12x + 27y + 19$
8. $f(x, y, z) = x^2 + (y-3)^2 + \left(z+1\right)^2$

Absolute Extrema.

With absolute extrema with one variable, we found our critical points, checked to see that they were on the interval we were interested in, and then also checked the endpoints of the interval. We put all these values into the original function and compared the values we got. We do something similar with two variables (or more), but we have to use the D-test to check the critical points. Our "intervals" are now regions in two dimensions. The regions are bounded by equations and may be described in words, or in sent notation with nonlinear inequalities. We will use these equations to reduce our original function to one variable, and test for one-dimensional critical points. And, we will have to test any places where our boundary functions meet. This process can be generalized to higher dimensions, but we will only be considering the two-variable case.

Example 3. Find the absolute extrema of the function $f(x,y)=3x^2+2y^2-4y$ on the region $R:\{(x,y)\mid y\ge x^2 \text{ and } y\le 4\}$.



The first thing to do is find any critical points.

$$f_x(x,y)=6x\Rightarrow x=0$$
$$f_y(x,y)=4y-4\Rightarrow 4y=4\Rightarrow y=1$$

So we find a critical point at (0,1). Is this in the region? Yes, so keep it. Incidentally, by the D-test, this is a minimum on the region, so what we are looking for on the boundary will be a maximum. If we had more than one point or a saddle point, we could not conclude this. We will check the result numerically at the end anyway.

Now we need to test the boundary conditions. One boundary is at the line y=4. Put this value into the f equation and reduce to just a function of x.

$$f(x,4)=3x^2+2(4)^2-4(4)=3x^2+32-16=3x^2+16$$

Taking the first derivative, we get $f'(x,4)=6x$, which has a critical point at x=0. So now we have the point (0,4) to test later.

The second boundary is $y=x^2$, so make this replacement.

$$f(x,x^2)=3x^2+2(x^2)^2-4(x^2)=3x^2+2x^4-4x^2=2x^4-x^2$$

Taking the first derivative, we get $f'(x,x^2)=8x^3-2x=2x(4x^2-1)$. This function has critical points at x=0: (0,0), and at $\pm\frac{1}{2}$: $\left(\frac{1}{2},\frac{1}{4}\right),\left(-\frac{1}{2},\frac{1}{4}\right)$. The graphs intersect at (2,4), and (-2,4). This gives us, finally, seven points to test in the original function.

$$f(0,1)=3(0)^2+2(1)^2-4(1)=2-4=-2$$
$$f(0,4)=3(0)^2+2(4)^2-4(4)=32-16=16$$
$$f(0,0)=3(0)^2+2(0)^2-4(0)=0$$
$$f\left(\frac{1}{2},\frac{1}{4}\right)=3\left(\frac{1}{2}\right)^2+2\left(\frac{1}{4}\right)^2-4\left(\frac{1}{4}\right)=-\frac{1}{8}$$
$$f\left(-\frac{1}{2},\frac{1}{4}\right)=3\left(-\frac{1}{2}\right)^2+2\left(\frac{1}{4}\right)^2-4\left(\frac{1}{4}\right)=-\frac{1}{8}$$
$$f(2,4)=3(2)^2+2(4)^2-4(4)=12+32-16=28$$
$$f(-2,4)=3(-2)^2+2(4)^2-4(4)=28$$

As expected, the minimum on the region at (0,1,-2) is the absolute minimum. And the absolute maximum on the region occurs at the intersections of the boundary conditions, (2,4,28) and (-2,4,28). Since the two values are the same, both are absolute maxima.

Practice Problems. Find the absolute extrema for each function on the indicated region.

9. $f(x,y)=x^2+xy, R:\{(x,y)\mid -2\le x\le 2,-1\le y\le 1\}$

10. $f(x, y) = 12 - 3x - 2y$ on the region bounded by the triangle with vertices (2,0), (0,1) and (1,2).
11. $f(x, y) = 2x - 2xy + y^2, R: \{(x, y) \mid y \geq x^2, y \leq 1\}$
12. $f(x, y) = x^2 + 2xy + y^2, R: \{(x, y) \mid x^2 + y^2 \leq 8\}$