∇ Notation

Multivariable calculus uses 2 special characters, both based on the Greek letter delta: ∇ and ∂. You can think of the former like the "uppercase" and the latter as the "lowercase" versions of the same character, and both are pronounced "del". ∂ is used in partial derivatives: $\frac{\partial z}{\partial x}$ vs. $\frac{dz}{dx}$ to indicate the former is a partial derivative where the second one is not.

The ∇ notation is considerably more complicated as an operator and it is used in several different ways. We'll go through each of these, one at a time, and look at what happens when we apply ∇ in coordinate systems other than rectangular coordinates. ∇ is a vector and so often you will see it written as $\vec{\nabla}$.

1. **The Gradient**

The ∇ notation is first encountered in calculating the gradient of a function.

∇ is short for a vector whose components are the operators for taking derivatives: $\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$. These are the partial derivative operators for each variable. In the gradient, this vector is used like a vector multiplied by a scalar. Consider the vector $< x, y, z >$ multiplied by the constant a: $< x, y, z > a =< xa, ya, za >$. Similarly, $\nabla f = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$. The resulting vector, therefore, is a vector whose components are the partial derivatives of the function.[1]

∇f is also sometimes written as just $\operatorname{grad} f$.

Let's look at a specific example.

Example 1. Find ∇f for $f(x, y, z) = 3x^2 + 2y^2 z$.

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \left\langle 6x, 4yz, 2y^2 \right\rangle$$

∇f takes a function and turns it into a vector. Make sure that's what you get.

[1] Note: we multiplied on the right in our example, even though it's non-standard for most vectors and constants because the order matters when working with operators, and this made the analogy a little less opaque.

What happens, though, if our function is in cylindrical (polar) or spherical coordinates?

What we can't do is work in rectangular, because not only must our coordinate values change, but the coordinates themselves change relationships to each other. I won't derive the gradient vector here, but I will give you the resulting formulas below. Our goal will be to apply them correctly.

In cylindrical coordinates: $\nabla = \left\langle \frac{\partial}{\partial r}, \frac{1}{r} \cdot \frac{\partial}{\partial \theta}, \frac{\partial}{\partial z} \right\rangle$

In spherical coordinates: $\nabla = \left\langle \frac{\partial}{\partial \rho}, \frac{1}{\rho} \cdot \frac{\partial}{\partial \varphi}, \frac{1}{\rho \sin\varphi} \cdot \frac{\partial}{\partial \theta} \right\rangle$

Let's see how different this makes things look by converting our function from example 1 into cylindrical and spherical coordinates and seeing what the gradient looks like. It should be noted that this is isn't going to be a pretty function in either system and so the gradient will be messy. In practice, you'll be applying these to functions that are simpler in cylindrical or spherical than in rectangular.

Example 2. Find ∇f for $f(x,y,z) = 3x^2 + 2y^2 z$ in cylindrical and spherical coordinates.

$$f(x,y,z) = 3x^2 + 2y^2 z$$
$$f(r,\theta,z) = 3r^2 \cos^2\theta + 2r^2 z \sin^2\theta$$
$$f(\rho,\varphi,\theta) = 3\rho^2 \sin^2\varphi \cos^2\theta + 2\rho^3 \sin^2\varphi \sin^2\theta \cos\varphi$$

In cylindrical then:

$$\frac{\partial f}{\partial r} = 6r\cos^2\theta + 4rz\sin^2\theta$$
$$\frac{1}{r}\left[\frac{\partial f}{\partial \theta}\right] = \frac{1}{r}\left[-6r^2\cos\theta\sin\theta + 4r^2 z\sin\theta\cos\theta\right] = -6r\cos\theta\sin\theta + 4rz\sin\theta\cos\theta$$
$$\frac{\partial f}{\partial z} = 2r^2\sin^2\theta$$
$$\nabla f_{cylindrical} = \left\langle 6r\cos^2\theta + 4rz\sin^2\theta, -6r\cos\theta\sin\theta + 4rz\sin\theta\cos\theta, 2r^2\sin^2\theta \right\rangle$$

Only the z coordinate in this example is what you'd expect from simply converting the rectangular gradient because the z direction is the only one that didn't change.

In spherical:

$$\frac{\partial f}{\partial \rho} = 6\rho\sin^2\varphi\cos^2\theta + 6\rho^2\sin^2\varphi\sin^2\theta\cos\varphi$$

$$\frac{1}{\rho}\left[\frac{\partial f}{\partial \varphi}\right] = \frac{1}{\rho}\left[6\rho^2\sin\varphi\cos\varphi\cos^2\theta + 4\rho^3\sin\varphi\cos\varphi\sin^2\theta\cos\varphi - 2\rho^3\sin^2\varphi\sin^2\theta\sin\varphi\right]$$

$$= 6\rho^2\sin\varphi\cos\varphi\cos^2\theta + 4\rho^2\sin\varphi\cos\varphi\sin^2\theta\cos\varphi - 2\rho^2\sin^3\varphi\sin^2\theta$$

$$\frac{1}{\rho\sin\varphi}\left[\frac{\partial f}{\partial \theta}\right] = \frac{1}{\rho\sin\varphi}\left[-6\rho^2\sin^2\varphi\cos\theta\sin\theta + 4\rho^3\sin^2\varphi\cos\varphi\sin\theta\cos\theta\right]$$

$$= -6\rho\sin\varphi\cos\theta\sin\theta + 4\rho^2\sin\varphi\cos\varphi\sin\theta\cos\theta$$

$$\nabla f_{spherical} = \left\langle \begin{array}{l} 6\rho\sin^2\varphi\cos^2\theta + 6\rho^2\sin^2\varphi\sin^2\theta\cos\varphi \\ 6\rho^2\sin\varphi\cos\varphi\cos^2\theta + 4\rho^2\sin\varphi\cos\varphi\sin^2\theta\cos\varphi - 2\rho^2\sin^3\varphi\sin^2\theta \\ -6\rho\sin\varphi\cos\theta\sin\theta + 4\rho^2\sin\varphi\cos\varphi\sin\theta\cos\theta \end{array} \right\rangle$$

Note the product rule that was necessary for the derivative with respect to φ, and I wrote the final result in vertical form because it's so long, it runs right off the side of the page if written horizontally.

There are certain fields, particularly in physics, where you will work almost exclusively in cylindrical or spherical coordinates because that's where the equations are simplest. The practice problems below contain functions in each of the three major coordinate systems for you to practice on. Apply the correct gradient formula to each problem. You should not need to convert systems for any problem.

Practice Problems.

a. Find the gradient, ∇f , for each function in the appropriate gradient formula.

1. $f(x,y,z) = xy^2 + x^2z + yz^2$
2. $f(x,y,z) = xy\cos z$
3. $f(x,y,z) = ze^{xy}$
4. $f(x,y,z) = \tan(x+y) + \tan(yz) - 1$
5. $f(x,y,z) = x\ln y + y^2z + z^2 - 8$
6. $f(x,y,z) = \sqrt{25 - 5x^2 - 5y^2}$
7. $f(x,y,z) = \dfrac{1}{\sqrt{1 - x^2 - y^2 - z^2}}$
8. $f(r,\theta,z) = r\csc\theta\cot\theta$
9. $f(r,\theta,z) = r^2\cos 2\theta + z^2 + 1$
10. $f(r,\theta,z) = r^2\cos^2\theta - z$
11. $f(r,\theta,z) = r^3z - \dfrac{6}{1 - r\cos\theta}$
12. $f(r,\theta,z) = re^{\theta} + z$

13. $f(\rho,\varphi,\theta)=4\rho\cos\varphi$
14. $f(\rho,\varphi,\theta)=3\rho\csc\varphi\sec\theta$
15. $f(\rho,\varphi,\theta)=\rho^2-2\rho\cos\varphi$
16. $f(\rho,\varphi,\theta)=\rho^2\sin^2\varphi+2\rho\tan\theta$

This handout is about the "how-to". Applications will be dealt with elsewhere.

2. The Curl

The ∇ is used in finding the curl of a vector as well. Here we write it as $\vec{\nabla}\times\vec{F}$, or sometimes you'll just see $\operatorname{curl}\vec{F}$. Since ∇ is a vector and here $\vec{F}$ is also, we calculate the curl the same way we do a cross product. Recall that for a vector $\vec{u}=\langle a,b,c\rangle$ and $\vec{v}=\langle x,y,z\rangle$, then $\vec{u}\times\vec{v}$ is given by the determinant of the matrix

$$\begin{vmatrix}\hat{i} & j & k\\ a & b & c\\ x & y & z\end{vmatrix}=(bz-cy)\hat{i}-(az-cx)j+(ay-bx)k.$$

Similarly, for $\vec{\nabla}\times\vec{F}$, where $\vec{F}=\langle M,N,P\rangle$, we have

$$\begin{vmatrix}\hat{i} & j & k\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ M & N & P\end{vmatrix}=\left(\frac{\partial P}{\partial y}-\frac{\partial N}{\partial z}\right)\hat{i}-\left(\frac{\partial P}{\partial x}-\frac{\partial M}{\partial z}\right)j+\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)k.$$

The result is another vector. For rectangular coordinates, this method is probably better than memorizing the formula that results. For cylindrical and spherical coordinates, if there is a nice way of memorizing the formulas for the curl, I don't know what it is.

In cylindrical, $\vec{\nabla}\times\vec{F}$ where $\vec{F}=\langle M,N,P\rangle$ (whose coordinates are (r,θ,z) coordinates), we get $\left\langle\left(\frac{1}{r}\cdot\frac{\partial P}{\partial\theta}-\frac{\partial N}{\partial z}\right),\left(\frac{\partial M}{\partial z}-\frac{\partial P}{\partial r}\right),\frac{1}{r}\cdot\left(\frac{\partial}{\partial r}[rN]-\frac{\partial M}{\partial\theta}\right)\right\rangle$.

In spherical, $\vec{\nabla}\times\vec{F}$ where $\vec{F}=\langle M,N,P\rangle$ (whose coordinates are (ρ,φ,θ) coordinates) we get $\left\langle\frac{1}{\rho\sin\varphi}\left(\frac{\partial}{\partial\varphi}[\sin\varphi N]-\frac{\partial P}{\partial\theta}\right),\frac{1}{\rho}\cdot\left(\frac{1}{\sin\varphi}\frac{\partial M}{\partial\theta}-\frac{\partial}{\partial\rho}[\rho N]\right),\frac{1}{\rho}\cdot\left(\frac{\partial}{\partial\rho}[\rho P]-\frac{\partial M}{\partial\varphi}\right)\right\rangle$.

Example 3. $\vec{F}(x,y,z)=\langle xy,yz,xz\rangle$. Find the curl.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & xz \end{vmatrix} = (0-y)\hat{i} - (z-0)\,j + (0-x)k = \langle -y, -z, -x \rangle$$

Example 4. Find $\vec{\nabla} \times \vec{F}$ for $\vec{F}(\rho,\varphi,\theta) = \langle \rho^2 \sin\varphi, \rho\cos\theta, \sin\varphi\cos\theta \rangle$.

Before we take the derivative in the first coordinate we need to multiply the N function by sinφ: $\sin\varphi(\rho\cos\theta) = \rho\sin\varphi\cos\theta$, and then take the derivative of that with respect to φ $\Rightarrow \rho\cos\varphi\cos\theta$. And we also need $\frac{\partial}{\partial\theta}(\sin\varphi\cos\theta) = -\sin\varphi\sin\theta$. Put these together to complete the first coordinate:

$$\frac{1}{\rho\sin\varphi}\left[\rho\cos\varphi\cos\theta + \sin\varphi\sin\theta\right] = \cot\varphi\cos\theta + \frac{\sin\theta}{\rho}$$

The second coordinate needs $\rho(\rho\cos\theta) = \rho^2\cos\theta$ and the derivative with respect to ρ of that $\Rightarrow 2\rho\cos\theta$. We also need $\frac{\partial}{\partial\theta}\left[\rho^2\sin\varphi\right] = 0$. Therefore, our second coordinate, if we put this all together is $\frac{1}{\rho}\left[0 - 2\rho\cos\theta\right] = -2\cos\theta$.

For the third coordinate, we need $\rho(\sin\varphi\cos\theta)$ and the derivative of this with respect to ρ $\Rightarrow \sin\varphi\cos\theta$, and we also need $\frac{\partial}{\partial\varphi}\left[\rho^2\sin\varphi\right] = \rho^2\cos\varphi$. Putting these together we get $\frac{1}{\rho}\left[\sin\varphi\cos\theta - \rho^2\cos\varphi\right] = \frac{1}{\rho}\sin\varphi\cos\theta - \rho\cos\varphi$.

Put these three components together to get the curl:

$$\vec{\nabla} \times \vec{F} = \left\langle \cot\varphi\cos\theta + \frac{\sin\theta}{\rho}, -2\cos\theta, \frac{1}{\rho}\sin\varphi\cos\theta - \rho\cos\varphi \right\rangle.$$

Practice Problems.

b. For each of the following functions, calculate the curl in the appropriate coordinate system.

17. $\vec{F}(x,y,z) = \langle xyz, x^2y, yz^2 \rangle$
18. $\vec{F}(x,y,z) = \langle \cos xy, \sin xz, \tan y \rangle$
19. $\vec{F}(x,y,z) = \left\langle \frac{yz}{\sqrt{1-x^2y^2}}, \frac{xz}{\sqrt{1-x^2y^2}}, \arcsin xy \right\rangle$

20. $\vec{F}(x,y)=\left\langle \frac{1}{x},\frac{1}{y},0\right\rangle$
21. $\vec{F}(r,\theta,z)=\left\langle r^2\sin\theta, r\sec\theta, z\right\rangle$
22. $\vec{F}(r,\theta,z)=\left\langle \tan z, \arctan r, \theta\right\rangle$
23. $\vec{F}(r,\theta,z)=\left\langle \ln r, r\cos z, z\tan\theta\right\rangle$
24. $\vec{F}(\rho,\varphi,\theta)=\left\langle \rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi\right\rangle$
25. $\vec{F}(\rho,\varphi,\theta)=\left\langle \rho^3, \theta\sin\varphi, \varphi^2\cos^2\theta\right\rangle$
26. $\vec{F}(\rho,\varphi,\theta)=\left\langle \frac{1}{\rho}, \rho^2\sin^3(\varphi\theta), \ln\rho\right\rangle$

3. The Divergence

The third major application of the ∇ notation is to calculate the divergence of a vector field. The divergence itself is a function and not a vector. What operation turns a vector into a number? The dot product. So the divergence is given by $\vec{\nabla}\cdot\vec{F}$ or sometimes just $\operatorname{div}\vec{F}$. For $\vec{F}=\langle M,N,P\rangle$, this gives $\left\langle \frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right\rangle\cdot\langle M,N,P\rangle=\frac{\partial M}{\partial x}+\frac{\partial N}{\partial y}+\frac{\partial P}{\partial z}$.

Of course, nothing is quite so simple in cylindrical or spherical coordinates are given below, respectively.

$$\vec{\nabla}\cdot\vec{F}=\frac{1}{r}\cdot\frac{\partial}{\partial r}\left[rM\right]+\frac{1}{r}\cdot\frac{\partial N}{\partial\theta}+\frac{\partial P}{\partial z}$$
$$\vec{\nabla}\cdot\vec{F}=\frac{1}{\rho^2}\cdot\frac{\partial}{\partial r}\left[\rho^2M\right]+\frac{1}{\rho\sin\varphi}\cdot\frac{\partial}{\partial\varphi}\left[\sin\varphi\cdot N\right]+\frac{1}{\rho\sin\varphi}\frac{\partial P}{\partial\theta}$$

Not pretty, but not as bad as the curl formulas.

Example 5. Find the divergence of $\vec{F}(x,y,z)=\left\langle yz,\cos xy, x^2y\right\rangle$.

$\vec{\nabla}\cdot\vec{F}=0+(-\sin xy\cdot x)+0=-x\sin xy$

Example 6. Find the divergence of $\vec{F}(\rho,\varphi,\theta)=\left\langle \rho^2\sin\varphi,\theta^2,\varphi\cos\theta\right\rangle$.

$$\vec{\nabla}\cdot\vec{F}=\frac{1}{\rho^2}\cdot\frac{\partial}{\partial r}\left[\rho^2\cdot\rho^2\sin\varphi\right]+\frac{1}{\rho\sin\varphi}\cdot\frac{\partial}{\partial\varphi}\left[\sin\varphi\cdot\theta^2\right]+\frac{1}{\rho\sin\varphi}\frac{\partial}{\partial\theta}\left[\varphi\cos\theta\right]$$

$$=\frac{1}{\rho^2}\cdot\frac{\partial}{\partial r}\left[\rho^4\sin\varphi\right]+\frac{1}{\rho\sin\varphi}\cdot\frac{\partial}{\partial\varphi}\left[\sin\varphi\cdot\theta^2\right]+\frac{1}{\rho\sin\varphi}\frac{\partial}{\partial\theta}\left[\varphi\cos\theta\right]$$

$$=\frac{1}{\rho^2}\left[4\rho^3\sin\varphi\right]+\frac{1}{\rho\sin\varphi}\left[\cos\varphi\cdot\theta^2\right]+\frac{1}{\rho\sin\varphi}\left[-\varphi\sin\theta\right]=4\rho\sin\varphi+\frac{\theta^2\cot\varphi}{\rho}-\frac{\varphi\sin\theta}{\rho\sin\varphi}$$

Practice Problems.

c. Find the divergence of each vector function in the appropriate coordinate system.

27. $\vec{F}(x,y,z)=\langle xyz, x^2y, yz^2\rangle$
28. $\vec{F}(x,y,z)=\langle \cos xy, \sin xz, \tan y\rangle$
29. $\vec{F}(x,y,z)=\left\langle \frac{yz}{\sqrt{1-x^2y^2}}, \frac{xz}{\sqrt{1-x^2y^2}}, \arcsin xy\right\rangle$
30. $\vec{F}(x,y)=\left\langle \frac{1}{x}, \frac{1}{y}, 0\right\rangle$
31. $\vec{F}(r,\theta,z)=\langle r^2\sin\theta, r\sec\theta, z\rangle$
32. $\vec{F}(r,\theta,z)=\langle \tan z, \arctan r, \theta\rangle$
33. $\vec{F}(r,\theta,z)=\langle \ln r, r\cos z, z\tan\theta\rangle$
34. $\vec{F}(\rho,\varphi,\theta)=\langle \rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi\rangle$
35. $\vec{F}(\rho,\varphi,\theta)=\langle \rho^3, \theta\sin\varphi, \varphi^2\cos^2\theta\rangle$
36. $\vec{F}(\rho,\varphi,\theta)=\left\langle \frac{1}{\rho}, \rho^2\sin^3(\varphi\theta), \ln\rho\right\rangle$

When dealing with vectors, we can also combine the cross product and the dot product into the triple scalar product. However, the divergence of the curl is always zero. In ∇ notation, $\nabla\cdot(\nabla\times F)=0$.

4. The Laplacian

There is a fourth way to use ∇ notation, which is to dot the ∇ with a gradient vector: $\nabla\cdot(\nabla f)$, or more compactly written $\nabla^2 f$. Just as with our other use of the dot product, it produces a function, here, the sum of the *second* derivatives.

$$\nabla^2 f=\nabla\cdot\nabla f=\left\langle\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right\rangle\cdot\left\langle\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right\rangle=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}.$$

This is called the Laplacian.

In cylindrical we have

$$\nabla^2 f = \frac{1}{r}\cdot\frac{\partial}{\partial r}\left[r\frac{\partial f}{\partial r}\right]+\frac{1}{r^2}\cdot\frac{\partial^2 f}{\partial\theta^2}+\frac{\partial^2 f}{\partial z^2}.$$

And in spherical we have

$$\nabla^2 f = \frac{1}{\rho^2}\cdot\frac{\partial}{\partial\rho}\left[\rho^2\frac{\partial f}{\partial\rho}\right]+\frac{1}{\rho^2\sin\varphi}\cdot\frac{\partial}{\partial\varphi}\left[\sin\varphi\cdot\frac{\partial f}{\partial\varphi}\right]+\frac{1}{\rho^2\sin^2\varphi}\cdot\frac{\partial^2 f}{\partial\theta^2}.$$

Example 7. Find the Laplacian of $f(x,y,z)=3x^2+2y^2z$.

$$\nabla^2 f = \frac{\partial}{\partial x}(6x)+\frac{\partial}{\partial y}(4yz)+\frac{\partial}{\partial z}(2y^2)=6+4z+0=6+4z$$

Example 8. Find the Laplacian of $f(\rho,\varphi,\theta)=\rho^2\cos\theta\sin\varphi$.

$$\frac{\partial f}{\partial\rho}=2\rho\cos\theta\sin\varphi\Rightarrow 2\rho^3\cos\theta\sin\varphi\Rightarrow 6\rho^2\cos\theta\sin\varphi\Rightarrow 6\cos\theta\sin\varphi$$

$$\frac{\partial f}{\partial\varphi}=\rho^2\cos\theta\cos\varphi\Rightarrow\rho^2\cos\theta\sin\varphi\cos\varphi\Rightarrow\rho^2\cos\theta\left(\cos^2\varphi-\sin^2\varphi\right)\Rightarrow$$

$$\frac{\cos\theta\left(\cos^2\varphi-\sin^2\varphi\right)}{\sin\varphi}=\cos\theta\left(\cot\varphi\cos\varphi-\sin\varphi\right)$$

$$\frac{\partial f}{\partial\theta}=-\rho^2\sin\theta\cos\varphi\Rightarrow-\rho^2\cos\theta\cos\varphi\Rightarrow-\frac{\cos\theta\cos\varphi}{\sin^2\varphi}=-\cos\theta\cot\varphi\csc\varphi$$

$$\nabla^2 f = 6\cos\theta\sin\varphi+\cos\theta\left(\cot\varphi\cos\varphi-\sin\varphi\right)-\cos\theta\cot\varphi\csc\varphi$$

Practice Problems.

d. Find the Laplacian of the following functions in the appropriate coordinate systems.

37. $f(x,y,z)=xy^2+x^2z+yz^2$
38. $f(x,y,z)=xy\cos z$
39. $f(x,y,z)=ze^{xy}$
40. $f(x,y,z)=\tan(x+y)+\tan(yz)-1$
41. $f(x,y,z)=x\ln y+y^2z+z^2-8$

42. $f(x,y,z)=\sqrt{25-5x^2-5y^2}$
43. $f(x,y,z)=\dfrac{1}{\sqrt{1-x^2-y^2-z^2}}$
44. $f(r,\theta,z)=r\csc\theta\cot\theta$
45. $f(r,\theta,z)=r^2\cos 2\theta+z^2+1$
46. $f(r,\theta,z)=r^2\cos^2\theta-z$
47. $f(r,\theta,z)=r^3z-\dfrac{6}{1-r\cos\theta}$
48. $f(r,\theta,z)=re^{\theta}+z$
49. $f(\rho,\varphi,\theta)=4\rho\cos\varphi$
50. $f(\rho,\varphi,\theta)=3\rho\csc\varphi\sec\theta$
51. $f(\rho,\varphi,\theta)=\rho^2-2\rho\cos\varphi$
52. $f(\rho,\varphi,\theta)=\rho^2\sin^2\varphi+2\rho\tan\theta$

Of course, this barely scratches the surface of ∇ notation. Because it is a derivative operator, it has its own set of product rules. I list them here.

i. $\nabla(fg)=f\nabla g+g\nabla f$

ii. $\nabla\left(\vec{A}\cdot\vec{B}\right)=\vec{A}\times\left(\nabla\times\vec{B}\right)+\vec{B}\times\left(\nabla\times\vec{A}\right)+\left(\vec{A}\cdot\nabla\right)\vec{B}+\left(\vec{B}\cdot\nabla\right)\vec{A}$

iii. $\nabla\cdot\left(f\vec{A}\right)=f\left(\nabla\cdot\vec{A}\right)+\vec{A}\cdot\nabla f$

iv. $\nabla\cdot\left(\vec{A}\times\vec{B}\right)=\vec{B}\cdot\left(\nabla\times\vec{A}\right)-\vec{A}\cdot\left(\nabla\times\vec{B}\right)$

v. $\nabla\times\left(f\vec{A}\right)=f\left(\nabla\times\vec{A}\right)-\vec{A}\times(\nabla f)$

vi. $\nabla\times\left(\vec{A}\times\vec{B}\right)=\left(\vec{B}\cdot\nabla\right)\vec{A}-\left(\vec{A}\cdot\nabla\right)\vec{B}+\vec{A}\left(\nabla\cdot\vec{B}\right)-\vec{B}\left(\nabla\cdot\vec{A}\right)$

vii. $\nabla\cdot\left(\nabla\times\vec{A}\right)=0$ (this is the "triple scalar product" mentioned above)

viii. $\nabla\times(\nabla f)=0$

ix. $\nabla\times\left(\nabla\times\vec{A}\right)=\nabla\left(\nabla\cdot\vec{A}\right)-\nabla^2\vec{A}$

Practice Problems.

e. For each of the product rules listed above, verify each rule with the given functions. Verify the rule by completing both sides of the expression and show that they are equal.

53. $f(x,y,z)=x+y+z,\quad g(x,y,z)=xyz$
$\vec{A}(x,y,z)=\langle x,y,z\rangle,\quad \vec{B}(x,y,z)=\langle y,z,x\rangle$

54. $f(r,\theta,z) = r^2\cos\theta + z, \quad g(r,\theta,z) = r + z\sin\theta$
$\vec{A}(r,\theta,z) = \langle r, \tan\theta, z\rangle, \quad \vec{B}(r,\theta,z) = \langle z, \ln r, e^{\theta}\rangle$

55. $f(\rho,\varphi,\theta) = \rho\cos\theta\sin\varphi, \quad g(\rho,\varphi,\theta) = \theta\sin\rho\tan\varphi$
$\vec{A}(\rho,\varphi,\theta) = \langle \rho, \theta, \varphi\rangle, \quad \vec{B}(\rho,\varphi,\theta) = \langle \rho^2, \rho\ln\varphi, \arctan\theta\rangle$