Chain Rule Key

1. a. $w = \sqrt{x^2+y^2} = (x^2+y^2)^{1/2}$ $\quad x = \cos(t)$, $y = e^t$

$\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt} + \frac{dw}{dy}\frac{dy}{dt}$

$\frac{dw}{dx} = \frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2x = \frac{x}{\sqrt{x^2+y^2}} = \frac{\cos t}{\sqrt{\cos^2 t + e^{2t}}}$

$\frac{dw}{dy} = \frac{y}{\sqrt{x^2+y^2}} = \frac{e^t}{\sqrt{\cos^2 t + e^{2t}}}$

$\frac{dx}{dt} = -\sin t \qquad \frac{dy}{dt} = e^t$

$\frac{dw}{dt} = \frac{-\cos t \sin t + e^{2t}}{\sqrt{\cos^2 t + e^{2t}}}$

b. $w = x\sin y \quad x = e^t, \ y = \pi - t$

$\frac{dw}{dx} = \sin y = \sin(\pi - t)$

$\frac{dw}{dy} = x\cos y = e^t \cos(\pi - t)$

$\frac{dx}{dt} = e^t$

$\frac{dy}{dt} = -1$

$\frac{dw}{dt} = e^t \sin(\pi - t) - e^t \cos(\pi - t)$

②

1c. $w = \cos(x-y)$ $\qquad x = t^2 \quad y = 1$

$\frac{dw}{dx} = -\sin(x-y)(1) = -\sin(t^2-1)$

$\frac{dw}{dy} = \sin(x-y) = \sin(t^2-1)$

$\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = 0$

$\frac{dw}{dt} = -2t\sin(t^2-1) + 0$

d. $w = xy^2 + x^2z + yz^2$, $x = t^2$ $y = \arccos t$ $z = e^{-2t}$

$\frac{dw}{dx} = y^2 + 2xz = \arccos^2 t + 2t^2e^{-2t}$

$\frac{dw}{dy} = 2xy + z^2 = 2t^2\arccos t + e^{-4t}$

$\frac{dw}{dz} = x^2 + 2yz = t^4 + 2\arccos t)\, e^{-2t}$

$\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = \frac{-1}{\sqrt{1-t^2}} \qquad \frac{dz}{dt} = -2e^{-2t}$

$\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt} + \frac{dw}{dy}\frac{dy}{dt} + \frac{dw}{dz}\frac{dz}{dt} =$

$(\arccos^2 t + 2t^2e^{-2t})2t + (2t^2\arccos t + e^{-4t})\left(\frac{-1}{\sqrt{1-t^2}}\right)$

$+ (t^4 + 2e^{-2t}\arccos t)(-2e^{-2t})$

#2 1b. $\frac{d^2w}{dt^2} = \frac{d}{dt}\left[e^t\left(\sin(\pi-t) - \cos(\pi-t)\right)\right]$

$e^t\left(\cancel{\sin(\pi-t)} - \cos(\pi-t)\right) + e^t\left(\cos(\pi-t)(-1) + \cancel{\sin(\pi-t)(-1)}\right)$

$= -2e^t\cos(\pi-t)$

1c. $\frac{d^2w}{dt^2} = -2\sin(t^2-1) - 2t\cos(t^2-1)2t$

$= -2\sin(t^2-1) - 4t(\cos(t^2-1))$

3. $w = x^2+y^2 \quad x = s+t \quad y = s-t$

$\frac{dw}{dx} = 2x = 2(s+t) = 2s+2t$

$\frac{dw}{dy} = 2y = 2(s-t) = 2s-2t$

$\frac{dx}{dt} = 1 \quad \frac{dx}{ds} = 1 \quad \frac{dy}{dt} = -1 \quad \frac{dy}{ds} = 1$

$\frac{dw}{dt} = (2s+2t)(1) + (2s-2t)(-1) = 4t$

$\frac{dw}{ds} = (2s+\cancel{2t})(1) + (2s-\cancel{2t})(1) = 4s$

6. $w = y^3 - 3x^2y \quad x = e^s \quad y = e^{t^2}$

$\frac{dw}{dx} = 6xy = 6e^{s+t^2}$

(4)

$$\frac{\partial w}{\partial y} = 3y^2 - 3x^2 = 3e^{2s} - 3e^{2t^2}$$

$$\frac{\partial x}{\partial t} = 0 \qquad \frac{\partial x}{\partial s} = e^s \qquad \frac{\partial y}{\partial t} = 2te^{t^2} \qquad \frac{\partial y}{\partial s} = 0$$

$$\frac{\partial w}{\partial t} = \cancel{6e^{s+t^2} \cdot (0)} + (3e^{2s} - 3e^{2t^2})(2te^{t^2})$$

$$= 6te^{2s+t^2} - 6te^{2t^4}$$

$$\frac{\partial w}{\partial s} = (6e^{s+t^2})e^s + \cancel{(3e^{2s} - 3e^{2t^2})(0)}$$

$$= 6e^{2s+t^2}$$

c. $w = xyz \qquad x = s+t, \; y = s-t, \; z = st^2$

$$\frac{\partial w}{\partial x} = yz = (s-t)st^2$$

$$\frac{\partial w}{\partial y} = xz = (s+t)st^2$$

$$\frac{\partial w}{\partial z} = xy = (s+t)(s-t) = s^2 - t^2$$

$$\frac{\partial x}{\partial t} = 1 \quad \frac{\partial x}{\partial s} = 1 \quad \frac{\partial y}{\partial t} = (-1) \quad \frac{\partial y}{\partial s} = 1 \quad \frac{\partial z}{\partial t} = 2st \quad \frac{\partial z}{\partial s} = t^2$$

$$\frac{\partial w}{\partial t} = (\cancel{s}-t)st^2(1) + ((\cancel{s}+t)st^2)(-1) + (s^2-t^2)(2st)$$

$$-st^3 - st^3 + 2s^3t - 2st^3 = 2s^3t - 4st^3$$

$$\frac{\partial w}{\partial s} = (s-\cancel{t})st^2(1) + (s \cancel{+t})st^2(1) + (s^2-t^2)t^2$$

$$s^2t^2 + s^2t^2 + s^2t^2 - t^4 = 3s^2t^2 - t^4$$

5

d. $w = ze^{xy}$ $x = s-t$ $y = s+t$, $z = st$

$\frac{\partial w}{\partial x} = yze^{xy} = (s+t)(st)\,e^{s^2-t^2}$

$\frac{\partial w}{\partial y} = xze^{xy} = (s-t)(st)\,e^{s^2-t^2}$

$\frac{\partial w}{\partial z} = e^{xy} = e^{s^2-t^2}$

$\frac{\partial x}{\partial t} = -1$ $\frac{\partial x}{\partial s} = 1$ $\frac{\partial y}{\partial t} = 1$ $\frac{\partial y}{\partial s} = +1$ $\frac{\partial z}{\partial t} = s$ $\frac{\partial z}{\partial s} = t$

$\frac{\partial w}{\partial t} = (s+t)(st)e^{s^2-t^2}(-1) + (s-t)(st)e^{s^2-t^2}(1) + se^{s^2-t^2}$

$(\cancel{-s^2t} - st^2 + \cancel{s^2t} - st^2 + s)\,e^{s^2-t^2}$

$(-2st^2 + s)\,e^{s^2-t^2}$

$\frac{\partial w}{\partial s} = (s+\cancel{t})(st)e^{s^2-t^2}(1) + (s-\cancel{t})st\,e^{s^2-t^2}(1) + te^{s^2-t^2}$

$(2s^2t + t)\,e^{s^2-t^2}$

4a. $w = x^2 - 2xy + y^2$ $x = r+\theta$ $y = r-\theta$

$\frac{\partial w}{\partial x} = 2x - 2y = 2(\cancel{r}+\theta) - 2(\cancel{r}-\theta) = +4\theta$

$\frac{\partial w}{\partial y} = -2x + 2y = \cancel{-2}(r+\theta) + \cancel{2}(r-\theta) = -4\theta$

$\frac{\partial x}{\partial r} = 1$ $\frac{\partial x}{\partial \theta} = 1$ $\frac{\partial y}{\partial r} = 1$ $\frac{\partial y}{\partial \theta} = -1$

$\frac{\partial w}{\partial r} = 4\theta(1) + -4\theta(1) = 0$

$\frac{\partial w}{\partial \theta} = 4\theta(1) + {}^{-}4\theta(-1) = 8\theta$

6

46. $w = \sqrt{25 - 5x^2 - 5y^2} = (25 - 5x^2 - 5y^2)^{1/2}$

$x = r\cos\theta \quad y = r\sin\theta$

$w = \frac{\partial w}{\partial x} = \frac{1}{2}(25 - 5x^2 - 5y^2)^{-1/2} \cdot -10x = \frac{-5x}{\sqrt{25 - 5x^2 - 5y^2}} =$

$\frac{-5r\cos\theta}{\sqrt{25 - 5r^2}}$

$\frac{\partial w}{\partial y} = \frac{-5y}{\sqrt{25 - 5x^2 - 5y^2}} = \frac{-5r\sin\theta}{\sqrt{25 - 5r^2}}$

$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial r} = \sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$

$\frac{\partial w}{\partial r} = \frac{-5r\cos\theta \cdot \cos\theta}{\sqrt{25 - 5r^2}} + \frac{-5r\sin\theta \cdot \sin\theta}{\sqrt{25 - 5r^2}} =$

$\frac{-5r\cancel{(\cos^2\theta + \sin^2\theta)}}{\sqrt{25 - 5r^2}} = \frac{-5r}{\sqrt{25 - 5r^2}}$

$\frac{\partial w}{\partial \theta} = \frac{-5r\cos\theta}{\sqrt{25 - 5r^2}} \cdot -r\sin\theta + \frac{-5r\sin\theta}{\sqrt{25 - 5r^2}} \cdot r\cos\theta = 0$