Tangents & Normals

We encounter problems involving tangents and normals several times throughout the course, so we will return to this handout a couple different times. In some cases we will be looking at tangent vectors (along a curve) together with a normal vector, and the binormal vector (together, they define a three-dimensional space). In later cases we will be considering tangent planes (to a surface), and normal vectors; in one case the surface will be defined using traditional coordinate systems (rectangular, cylindrical or spherical), and in the second case, the surface will be defined parametrically. Each of these cases is a bit different, but they also have certain similarities, such as the terminology, and can be easily confused, especially once we have to consider all of them.

Parametric Curves in 3-space.
Given a parameterized curve, we want to know what the tangent vector to the curve is, what direction is it pointing in at any given instant along the path. Finding this vector is also useful for setting up a coordinate system that changes as the particle moved along the path.

Example 1. Consider the parameterized curve $\vec{r}(t)=3t\vec{i}-t\vec{j}+t^2\vec{k}$, find the unit tangent vector to the curve, the unit normal to the curve, and the unit binormal.

The first step is to find $\vec{r}\,'(t)=3\vec{i}-\vec{j}+2t\vec{k}$. This vector gives us both the direction, and a magnitude. However, we want an unit tangent vector, so we need to calculate the length of this vector and divide by it.

$$\vec{T}(t)=\frac{\vec{r}\,'(t)}{\left\|\vec{r}\,'(t)\right\|}=\frac{3\vec{i}-\vec{j}+2t\vec{k}}{\sqrt{10+4t^2}}$$

The first step isn't so bad, but the second step, for a problem like this one can be problematic because the unit normal is derived from the unit tangent vector, not just the acceleration vector.

$$\vec{N}(t)=\frac{\vec{T}\,'(t)}{\left\|\vec{T}\,'(t)\right\|}$$

In this example, that means we are going to need the chain rule or the quotient rule to work this out. I'll use the chain rule since it means I won't have to find a common denominator later.

$$\vec{T}(t) = \left(10+4t^2\right)^{-1/2}\left(3\vec{i} - \vec{j} + 2t\vec{k}\right)$$

$$\vec{T}'(t) = -\frac{1}{2}\left(10+4t^2\right)^{-3/2}(8t)\left(3\vec{i} - \vec{j} + 2t\vec{k}\right) + \left(10+4t^2\right)^{-1/2}\left(2\vec{k}\right) = \frac{-4t\left(3\vec{i} - \vec{j} + 2t\vec{k}\right)}{\left(10+4t^2\right)^{3/2}} + \frac{\left(10+4t^2\right)\left(2\vec{k}\right)}{\left(10+4t^2\right)^{3/2}}$$

$$= \frac{\left(-12t\vec{i} + 4t\vec{j} - 8t^2\vec{k}\right) + \left(20+8t^2\right)\vec{k}}{\left(10+4t^2\right)^{3/2}} = \frac{-12t\vec{i} + 4t\vec{j} + 20\vec{k}}{\left(10+4t^2\right)^{3/2}}$$

We next need the magnitude of this vector.

$$\vec{T}'(t) = \frac{-12t\vec{i} + 4t\vec{j} + 20\vec{k}}{\left(10+4t^2\right)^{3/2}}$$

$$\left\|\vec{T}'(t)\right\| = \frac{\sqrt{144t^2 + 16t^2 + 400}}{\left(10+4t^2\right)^{3/2}} = \frac{\sqrt{160t^2 + 400}}{\left(10+4t^2\right)^{3/2}} = \frac{4\sqrt{10t^2 + 25}}{\left(10+4t^2\right)^{3/2}}$$

Multiply the former by the reciprocal of the latter to get the unit normal vector. You notice that some parts cancel out.

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\left\|\vec{T}'(t)\right\|} = \frac{-12t\vec{i} + 4t\vec{j} + 20\vec{k}}{\left(10+4t^2\right)^{3/2}} \cdot \frac{\left(10+4t^2\right)^{3/2}}{4\sqrt{10t^2+25}} = \frac{-12t\vec{i} + 4t\vec{j} + 20\vec{k}}{4\sqrt{10t^2+25}} = \frac{-3t\vec{i} + t\vec{j} + 5\vec{k}}{\sqrt{10t^2+25}}$$

You can check for yourself that the vector is a unit vector now. The magnitude of the numerator is the denominator. But you see how the normalizing interacted with the result. This would not have been the answer obtained from just normalizing the acceleration. We can only get away with that is the magnitude of the tangent vector is constant.

Now, lastly the binormal vector: at least we don't have to take any more derivatives. The binormal is a vector perpendicular to both and so is defined as $\vec{B} = \vec{T} \times \vec{N}$. We will use properties of the cross product to pull out the normalizing factors.

$$\vec{B} = \vec{T} \times \vec{N} = \left(10+4t^2\right)^{-1/2}\left(10t^2+25\right)^{-1/2}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 2t \\ -3t & t & 5 \end{vmatrix} = \frac{\left(-5-2t^2\right)\vec{i} - \left(15+6t^2\right)\vec{j} + \left(3t-3t\right)\vec{k}}{\sqrt{\left(10+4t^2\right)\left(10t^2+25\right)}}$$

$$= \frac{-\left(2t^2+5\right)\vec{i} - \left(6t^2+15\right)\vec{j}}{\sqrt{40t^4 + 200t^2 + 250}}$$

A little bit of algebra will show that this vector is also a unit vector since the denominator is the magnitude of the numerator.

If we are given at point to analyze the function at, we must do this after the equation has been obtained. If you are given a point in rectangular coordinates rather than in terms of t, you will need to use the original parameterization to find the t value.

Remember, everything we have done in this example is to produce unit vectors. In the next examples we will be working with some vectors, but also some planes, and we will not need to use unit vectors. What happens next with surfaces is quite different from what happens with curves. Find a way to separate the two in your mind.

When we intersect two surfaces, we get a curve in 3-space. We can also use this technique to find the tangent vector to the curve of intersection, once we've parameterized the intersection.

Practice Problems. Find the unit tangent and unit normal vectors for the curves at the given point. If the magnitude of the tangent and normal vectors are constant, also find the binormal vector.

1. $\vec{r}(t) = 6\cos t\vec{i} + 2\sin t\vec{j}, t = \frac{\pi}{3}$
2. $\vec{r}(t) = 2\sin t\vec{i} + 2\cos t\vec{j} + 4\vec{k}, P\left(\sqrt{2}, \sqrt{2}, 4\right)$
3. $\vec{r}(t) = \ln t\vec{i} + (t+1)\vec{j}, t = 2$
4. $\vec{r}(t) = \left(t^3 - 4t\right)\vec{i} + 2t^2\vec{j}, t = 1$
5. $\vec{r}(t) = 4t\vec{i} - 4t\vec{j} + 2t\vec{k}, t = 2$

Tangent Planes to a Surface, and Normal Lines in 3-space.
We need to recall how to write the equation of a plane in 3-space. To do this we need a vector normal to the plane, say, <a,b,c>, and a point in the plane, say (h,k,l). The equation of the plane then is: $a(x-h)+b(y-k)+c(z-l)=0$.

The equation of line is done similarly, but with notable differences. Using the same vector and a point on the line, the symmetric form of the line is $\frac{x-h}{a} = \frac{y-k}{b} = \frac{z-l}{c}$.

To do this for a surface, we will replace the vector <a,b,c> with the gradient of the equation of the surface, $gradF = \nabla F(x,y,z)$, evaluated at the point.

Example 2. Find the tangent plane to the graph $x^2 + 3y + z^3 = 9$ at the point (2,-1,2). Find the normal line at the same point.

First, we need to find the master F function, by putting all the variables and constants on one side of the equation.

$$F(x,y,z) = x^2 + 3y + z^3 - 9$$

Next we need some partial derivatives. And then evaluate it at the given point.

$$\nabla F(x,y,z)=\frac{\partial F}{\partial x}\vec{i}+\frac{\partial F}{\partial y}\vec{j}+\frac{\partial F}{\partial z}\vec{k}=2x\vec{i}+3\vec{j}+3z^2\vec{k}$$

$$\nabla F(2,-1,2)=4\vec{i}+3\vec{j}+12\vec{k}$$

Now combine this vector with our point to form the tangent plane.

$$4(x-2)+3(y+1)+12(z-2)=0$$

You can distribute and collect the constants if you like, or solve for z, but that's not necessary unless you want to graph it.

Lastly, we need the normal line. Using the symmetric form of a line we get:

$$\frac{x-2}{4}=\frac{y+1}{3}=\frac{z-2}{12}$$

We can look at the graph and tangent plane together.

Practice Problems. Find the equations of the tangent plane and the normal line to the curve at the given point.

6. $x^2+4y^2+z^2=36, P(2,-2,4)$
7. $x^2+y^2+z=9, P(1,2,4)$
8. $y\ln xz^2=2, P(e,2,1)$
9. $xyz=10, P(1,2,5)$
10. $2xy-z^3=0, P(2,2,2)$
11. What would be the condition needed to make the tangent plane to the graph $z=3x^2+2y^2-3x+4y-5$ horizontal? Find that point. [Hint: when a plane is horizontal, what is the normal to the plane?]

Tangent Planes to a Surface, and Normal Lines to Parametric Surfaces in 3-space.
Parameterized surfaces in 3-space have 2 parametric variables, so we need some other method for finding tangent planes and normal vectors. If our parametric variables are u and v, we find the normal vector to the surface by the formula $\vec{r_u}\times\vec{r_v}$ where these component vectors are the partial derivatives of our parameterized surface, instead of using the gradient.

Example 3. Find the tangent plane to the graph $\vec{r}(u,v)=2u\cos v\vec{i}+3u\sin v\vec{j}+u^2\vec{k}$ at the point (0,6,4), and then find the equation of the normal line at the same point.

Start by taking the partial derivatives with respect to u and v of our surface.

$\vec{r_u}(u,v) = 2\cos v\vec{i} + 3\sin v\vec{j} + 2u\vec{k}$
$\vec{r_v}(u,v) = -2u\sin v\vec{i} + 3u\cos v\vec{j} + 0\vec{k}$

Next we need the cross product of these two vectors.

$$\vec{r_u}(u,v)\times\vec{r_v}(u,v) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2\cos v & 3\sin v & 2u \\ -2u\sin v & 3u\cos v & 0 \end{vmatrix}$$
$$= \left(0-6u^2\cos v\right)\vec{i} - \left(0+4u^2\sin v\right)\vec{j} + \left(6u\cos^2 v + 6u\sin^2 v\right)\vec{k}$$
$$= -6u^2\cos v\vec{i} - 4u^2\sin v\vec{j} + 6u\vec{k}$$

In order to get the vector at the point, we need to find out what u and v are at the point (0,6,4). We do this using the original parameterization.

$x = 0 = 2u\cos v$

$y = 6 = 3u\sin v \Rightarrow \sin v = 1 \Rightarrow v = \dfrac{\pi}{2}$

$z = 4 = u^2 \Rightarrow u = 2$

Using the data for z, we get two possible values. I am only considering one of them. Combining this with the information for y, we find a value for v. We get the same information from x, since u is non-zero, cos*v*=0. We can get a second parameterized point at $\left(-2, \dfrac{3\pi}{2}\right)$, but we just need one to obtain a vector.

Our normal vector is then $-6(2)^2\cos\frac{\pi}{2}\vec{i} - 4(2)^2\sin\frac{\pi}{2}\vec{j} + 6(2)\vec{k} = 0\vec{i} - 16\vec{j} + 12\vec{k}$. Using this information together with the point (0,6,4), we can obtain both the tangent plane equation and the normal line. (I use parametric form here, since we can't divide by zero in the symmetric form.)

$0(x-0) - 16(y-6) + 12(z-4) = 0 \Rightarrow 12(z-4) = 16(y-6)$
$\vec{r}(t) = 0\vec{i} + (6-16t)\vec{j} + (4+12t)\vec{k}$

Practice Problems. Find the tangent plane and normal line to the graph at the given point.

12. $\vec{r}(u,v) = u\vec{i} + v\vec{j} + uv\vec{k}, P(1,1,1)$
13. $\vec{r}(u,v) = 2\cos u\vec{i} + v\vec{j} + 2\sin u\vec{k}, P(2,4,0)$
14. $\vec{r}(u,v) = 3\cos v\cos u\vec{i} + 3\cos v\sin u\vec{j} + 5\sin v\vec{k}, P(3,0,0)$
15. $\vec{r}(u,v) = u\vec{i} + \frac{1}{4}v^3\vec{j} + v\vec{k}, P(-1,2,2)$

Orientable Surfaces.

When we begin to deal with surface integrals, we now need to consider one additional factor we haven't considered before and that is the orientation of the normal vector to a surface. When dealing with surface integrals, we'll need to consider whether the normal vector is pointing inward or outward. These vectors differ by a negative sign. In the past, the sign hasn't mattered, but now it will since it determines the direction of the flow.

If you are given an explicit function of z(x,y), for instance $z = 3x - y^2$, then we will form a function of 3 variables for determining the gradient by $G(x, y, z) = z - z(x, y)$, or in this case $G(x, y, z) = z - 3x + y^2$. (Note that the z coefficient is positive for this orientation.) This construction will allow us to point the gradient vector ∇G, which as we saw in the second section, away from the curve of the graph. If we wish the normal vector to point inward to the curve of the graph, then simply apply a negative to the gradient, i.e. $-\nabla G$.

To check the orientation of your normal vector, you can choose coordinates on the surface (that satisfies the original equation), and compare the vector obtained to the graph of the surface.

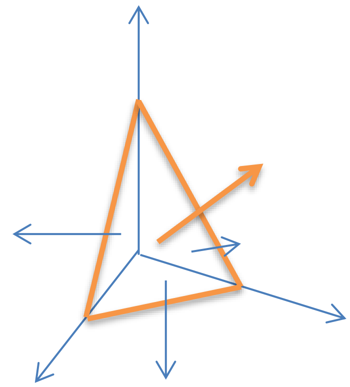
To calculate the normal vector for an implicit function, you may need to check the orientation in the region that you are working with, particularly if there is more than a single z term, but, typically, as with the above, allow the dominant z term to be positive.

If your surface is given in parametric form, typically the standard method for calculating the normal vector produced a vector oriented in the outward direction, so our methods from section three do not need to be tweaked, but there will be occasions where you may wish to calculate the flow in the opposite direction. The order of the cross product does matter, since switching $\vec{r_u} \times \vec{r_v}$ into $\vec{r_v} \times \vec{r_u}$ changes the orientation, and you can expect similar results from swapping the roles of the variables. So if you are getting a negative sign you didn't expect, check the orientation at a point you know (somewhere where the gradient is not zero).

Example 4. Consider calculating the normal vectors to the solid (one for each surface) bounded by the coordinate plane and the equation $z = 1 - x - y$.

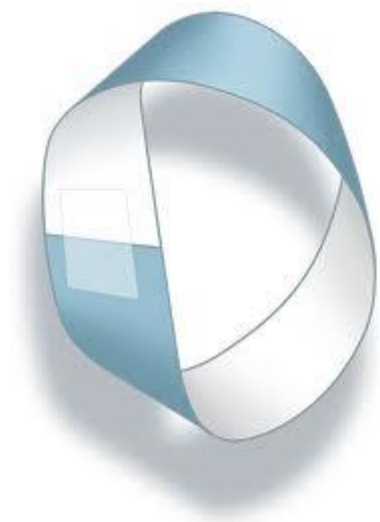
Another way of stating the plane surface $z = 1 - x - y$ is $x + y + z - 1 = 0$. This last form is our $G(x, y, z) = x + y + z - 1$, and $\nabla G = 1\hat{\imath} + 1\hat{\jmath} + 1\hat{k}$ is the normal vector to the surface. The graph of our surface is a plane with intercepts at (1,00), (0,1,0), and (0,0,1). So with the coordinate planes it forms a kind of pyramid with the peak of the pyramid at the origin. To orient the normal vector out of the surface, we want the vector pointing away from the origin, and in the first octant, that means all the components of the vector must be positive. (We can make this a unit normal by rescaling it by the length, i.e. $\vec{N} = \frac{\nabla G}{\|\nabla G\|} = \frac{1}{\sqrt{3}}\hat{\imath} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}\hat{k}$.

But how are we do deal with the coordinate planes? The normal vector to the xy-plane is either $\hat{k}\ or - \hat{k}$. Here, we don't want this vector to point up because that will point into the solid, so we choose $-\hat{k}$. Similarly for the yz-plane. The normal vector is either $\hat{\imath}\ or - \hat{\imath}$, and for the xz-plane it's either $\hat{\jmath}\ or - \hat{\jmath}$. In both cases we choose the negative vector since one the coordinate planes we want to point away from the solid, and thus away from the first octant.



It should be noted that not all surfaces are orientable.

The classic example of a non-orientable surface is a Möbius' strip. An image of one is below, but you can make your own by taking a strip of ribbon, turn one end over so that it's upside down relative to the other end, and then join the two ends together in a loop. You'll notice if you tried to take you finger and travel along the outward pointing surface, when you came back around to the point you started at, your finger would be on the inward facing surface.



You can create different versions of Möbius' strips by introducing more twists into the ribbon and repeating the experiment. These types of graphs are especially complex and are beyond the scope of this course.

Practice Problems. For each of the following situations, find the unit normal vector to the surface oriented outward and oriented inward. (Label these clearly, both which surface the vector belongs to and what its orientation is.) If the region is defined by more than one surface, find a set of unit normal vector for the outward orientation (as in Example 4) and the inward orientation.

16. Surface: $x^2 + y^2 + z^2 = 36$
17. Surface: $x^2 + y^2 + z^2 = 36$, first octant
18. Surface: $z = 1 - x^2 - y^2, z \geq 0$
19. Surface: $z = x^2 + y^2, x^2 + y^2 \leq 4, z = -1$
20. Surface: $z = 6 - 3x - 2y$, first octant