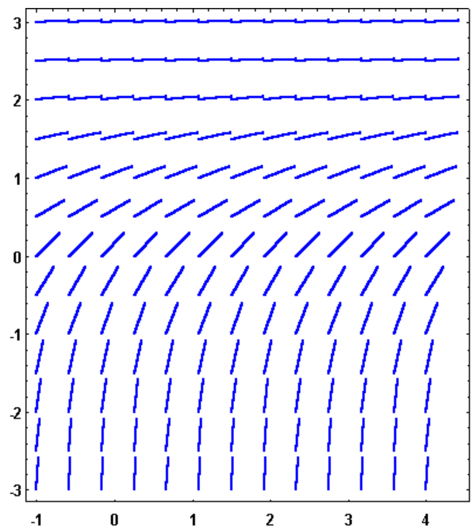
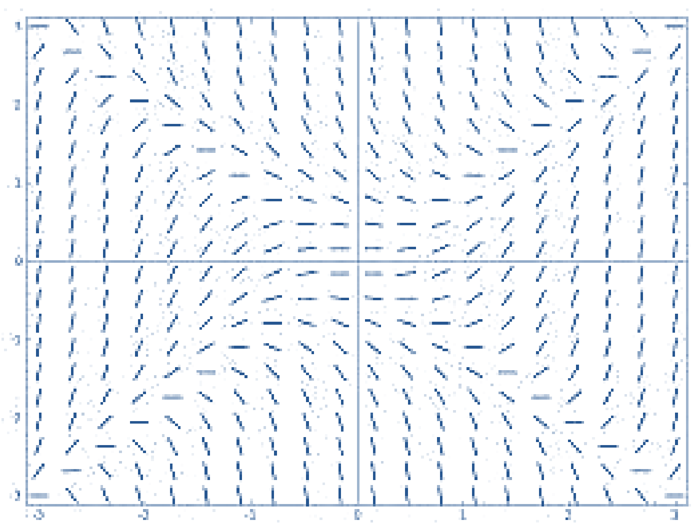
Graphing Direction (Slope) Fields

Direction fields are a type of vector field that shows how a differential equation behaves locally at some point in the plain. Since the differential equation contains the first derivative, which is itself representative of the slope of the curve at that point, direction fields are also called **slope fields**. We will discuss examples in this handout of graphing first order differential equations' direction fields and what we can say about the behaviour of solutions to the equation based on those fields. Explicit solutions to the equations will be done elsewhere.

Some examples of slope fields are shown below. The one on the left is the simplest kind that we will begin with, and autonomous equation that has only a y' and y, and no dependent variable. A more complicated example that depends on both y and x (or t) is shown on the right.



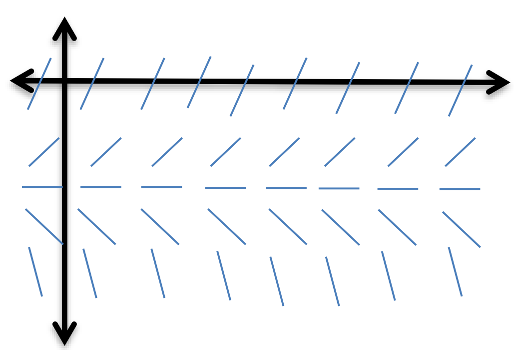


Let's start with an autonomous equation, of the form $y' = f(y)$.

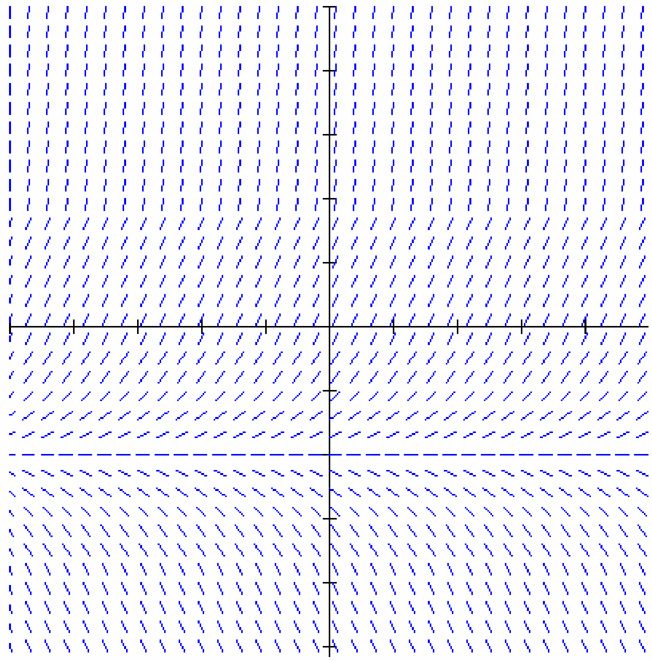
Example 1.
Plot the direction field for the differential equation $y' = y + 2$.

Autonomous equations are simple to plot direction fields by hand because there is no x (or t) dependency. When we choose a value for y to find the slope at that point, all the x-values along that horizontal line with fixed y have the same slope.

So for instance, if I plug in $y = 0$, I get a slope $\frac{dy}{dx} = 2$. If I plug in $y = -1$, then $\frac{dy}{dx} = 1$. When $y = -2$, then $\frac{dy}{dx} = 0$. And so forth. For each of these values, we're associated with a slope. We represent the slope at that point with a short line, usually of fixed length, but pointing in the direction a line with same slope would be in.

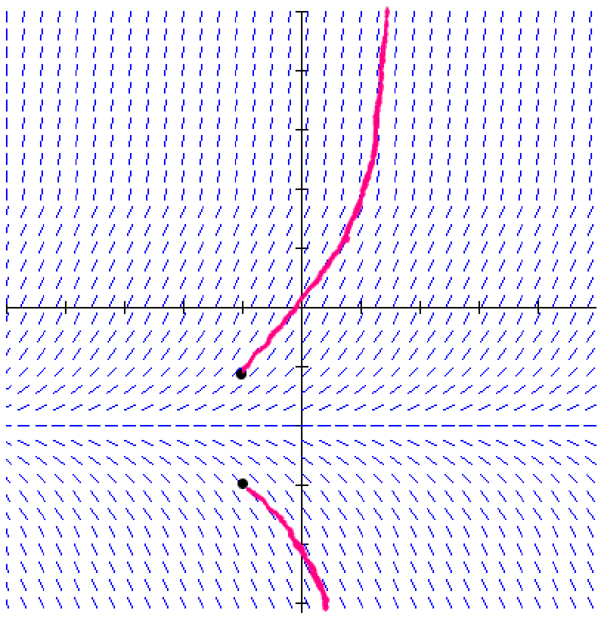


I've plotted these slopes on a small section of graph. A larger graph obtained from a Java applet is shown below.



How do we interpret a direction field? Suppose that we begin at any point in the plane. These are the initial conditions. The slope field tells us where to go next.

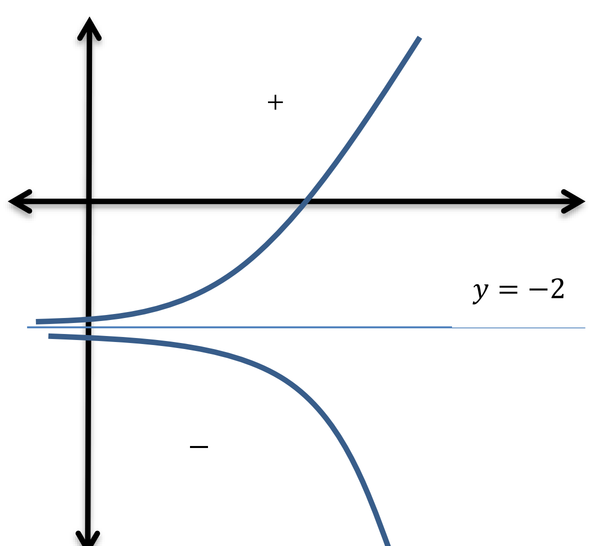
On this graph, compare the results of starting at the point (-1,-1) vs the point (-1,-3).



The direction field pushes the put upward from (-1,-1), increasing the value of y. But from (-1,-3), the direction field pushes the value of y down. Indeed, if we trace the graph backwards from these starting points, the curves will approach the line at $y = -2$ with the horizontal slope.

The most important feature of any direction field is any location where the slope is horizontal. These are the values for which the differential equation is equal to zero. This is called the **equilibrium solution**.

For a very general idea of the behavior of a graph, particularly if the differential equation is autonomous, we need only find the equilibrium solution(s) and the sign of the slope on either side of it. For instance, if we set this differential equation equal to zero. $0 = y + 2$, or $y = -2$. This is the location of the horizontal slopes on our graph. Then we can test the sign of the slope on either side. We showed above that the slope is positive. The slope below $y = -2$, say, at $y = -3$, then $\frac{dy}{dx} = -1$, and as we can see from the direction field graphed here, the slope is negative for all y-values below that as well. Consider the information summarized in the following graph.



The graphs approach the line, but cannot cross it, because the slopes remain positive or negative, but approach zero. It gives the motion of a particle in this direction field an exponential-like graph.

This differential equation is easy enough to solve for the equation explicitly and see that we do indeed obtain an exponential equation. This technique will be most useful, however, for equations that are more difficult to solve, but we are still interested in obtaining information about how the solution might behave.

Furthermore, equilibrium solutions (also called **steady state solutions**) can behave in a number of different ways. In Example 1, the equilibrium solution is a **repeller**, or an **unstable solution**, because while the solution is stable if you are exactly on the solution, if you are either to one side or the other, even by a small amount, the solution curve will move away from the equilibrium over time. This occurs when the slope is positive above the equilibrium line, and negative below it.

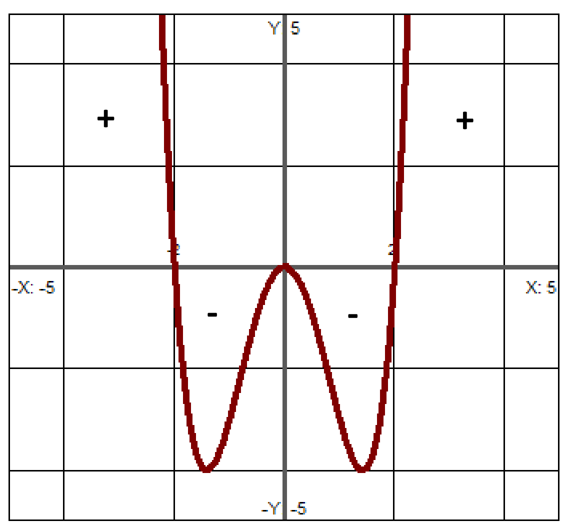
Stable solutions, or **attractors**, occur in the opposite situation, when the slope above the line is negative, and below it is positive. This causes the solution curves to get closer to the equilibrium over time, so even if you don't start out exactly on the stable solution, you will remain nearby (and even get closer) with time.

One can also have a **semi-stable solution**, or a **saddle point**. This happens when the slope of the line does not change across the equilibrium line, so that solutions on one side will approach the stable solution, but on the other side will move away from it.

Before we try a non-autonomous equation, let's try another autonomous one with a more complex direction field.

Example 2.
Sketch the direction field and its major characteristics such as any equilibrium solutions, and state the type of equilibrium solution each is for the differential equation $\frac{dy}{dx} = y^2(y^2 - 4)$.

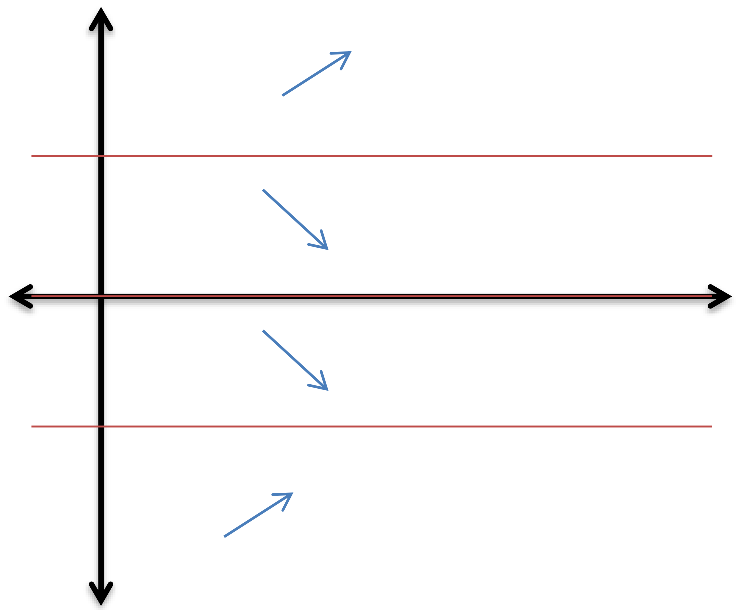
We are going to begin by setting the derivative to zero and search for equilibria. This particular equation is easy to solve for since it's partially factored already, and can easily be factored further to find the zeroes. However, since this isn't always the case, we are going to examine another method: drawing the phase plane. The phase plane is a graph that we can construct in two dimensions only when the differential equation is autonomous. The y variable will go on the horizontal axis, and the vertical axis will be $\frac{dy}{dx}$. We obtain the graph shown below.



From this graph it's possible to obtain the zeroes, even if they are irrational, or to see clearly if there are any complex zeroes we can ignore, and how many real zeroes to solve for.

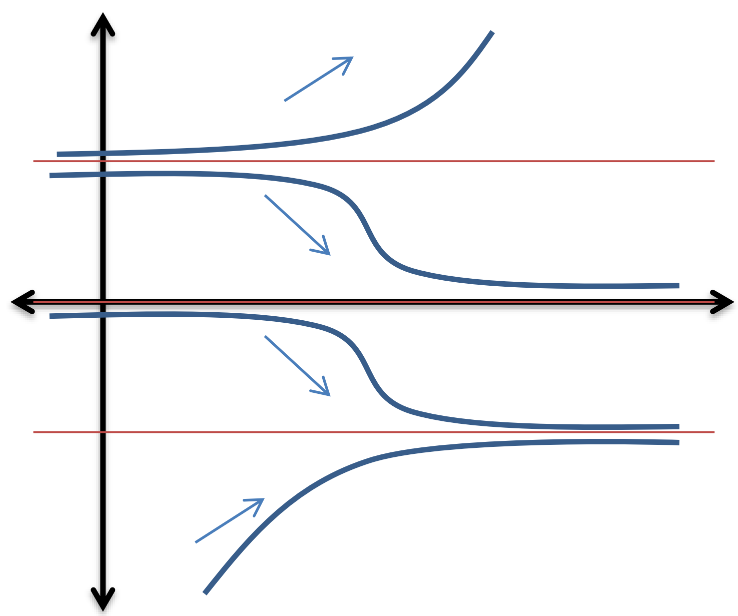
In addition, we can also obtain from this graph sign information about where the derivative is positive and where the derivative is negative.

We see from the graph that the zeroes are at $-2, 0, 2$, the same that we get from the algebra. We also can see that zero is a repeated root and a turning point: the sign on either side of this zero is not going to change. With practice, we'll be able to determine the stability of each equilibrium just by looking at the phase plane. But let's look at the direction field to verify this. We don't need a lot of detail to see what is going on. We should graph the horizontal lines representing each value of the steady state solutions, and then label the signs in each of the regions of the graph on either side of each one.



The slopes are positive when values of y are less than -2. The slopes are also positive when y is greater than 2. In between, and on both sides of 0, the slopes are negative. This agrees with the phase plane graph, and this is also the result we obtain when we check signs in the original equation.

What does this do to the behaviour of solutions? Outside -2 and 2, the solutions look quite similar to what we have seen before. But between -2 and 0, and again between 0 and 2, the solutions have the shape of logistic curves. They are exponential-like very close to one stable solution, and then they flip over, and are approximately exponential again as the approach the other stable solution. Examples of trajectories are shown below.

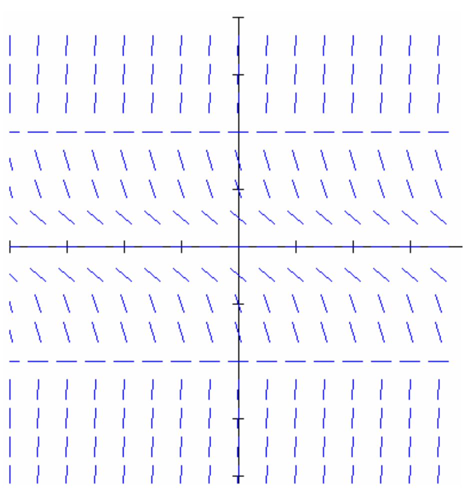


Using the terminology above, you'll note that the line $y = 2$ is an unstable solution since all the solution lines are moving away from the line.

$y = 0$ is a semi-stable solution, since some of the trajectories are moving toward it, but some are moving away from it (this is because of the lack of sign change across the equilibrium).

The $y = -2$ line is a stable equilibrium since solutions on both sides the line approach the line.

The graph of the complete direction field is shown.



Example 3.

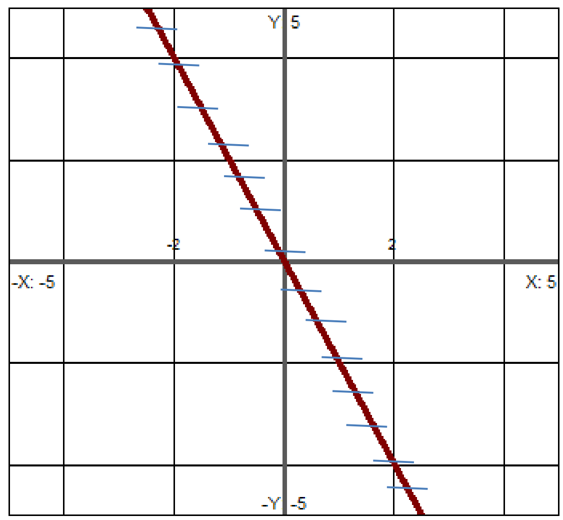
Sketch the direction field and some exemplar solution trajectories for the differential equation $\frac{dy}{dt} = -\frac{2t+y}{2y}$.

Differential equations that involve both y and the dependent variable are more difficult to graph the direction field for, but as before, we can begin with the steady state solution(s). Set $\frac{dy}{dt} = 0$, and solve for y.

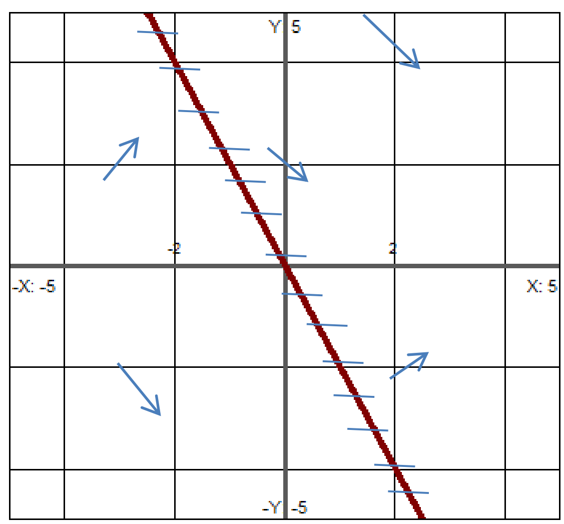
$$-\frac{2t+y}{2y} = 0$$
$$2t + y = 0$$
$$y = -2t$$

We must first graph this line. The slope along this entire line is zero.

The solutions, whatever they are, are either going to approach this line, or they are going to be repelled from this line. To determine what is going on, we are going to have to find the slope of the field at some sample points. It is much more effective to plot the field at all points in the plane (and we will let software complete that graph for us at the end of the problem), but we want to see how much we can determine with as little computation as possible. Let's choose the following test points: $(-3,2), (-3,-2), (0,2), (2,5), (2,-2)$.

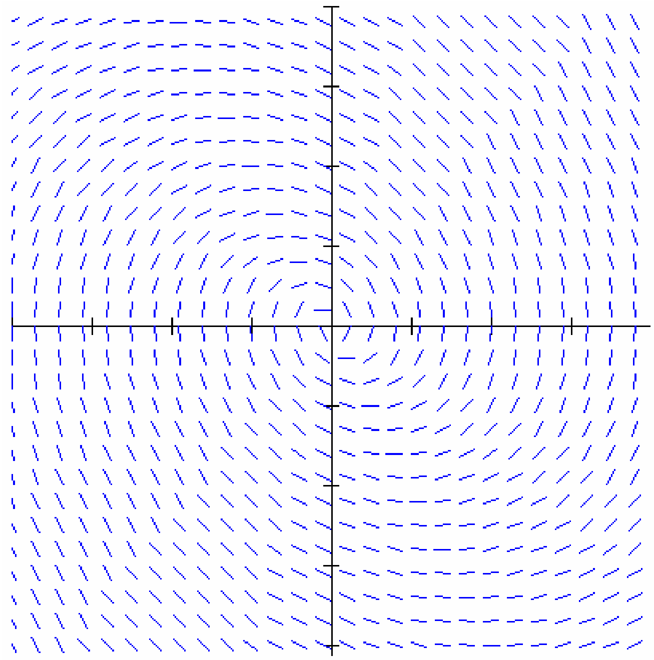


$$\frac{dy}{dt}(-3,2) = -\frac{2(-3)+2}{2(2)} = 1$$
$$\frac{dy}{dt}(-3,-2) = -\frac{2(-3)-2}{2(-2)} = -2$$
$$\frac{dy}{dt}(0,2) = -\frac{2(0)+2}{2(2)} = -\frac{1}{2}$$
$$\frac{dy}{dt}(2,5) = -\frac{2(2)+5}{2(5)} = -\frac{9}{10}$$
$$\frac{dy}{dt}(2,-2) = -\frac{2(2)-2}{2(-2)} = \frac{1}{2}$$



I've plotted the slopes on the graph. One the left side of the line it rather appears that the slopes may be approaching the equilibrium line, but on the right side, they appear to be moving away. Moreover, there is an apparent circular shape to the graph. Part of what makes this graph complicated is that there are vertical slopes at y=0. These are points where the solution curve breaks down and becomes undefined. We can plot trajectories up to these points, but since the horizontal axis represents time when we use t instead of x for the dependent variable, we cannot go back in time, and so cannot go leftward on the graph. If we use x, and this is a physical plane, we will end up with the swirling trajectory, since we will be able to go left.

The full direction field is plotted below.



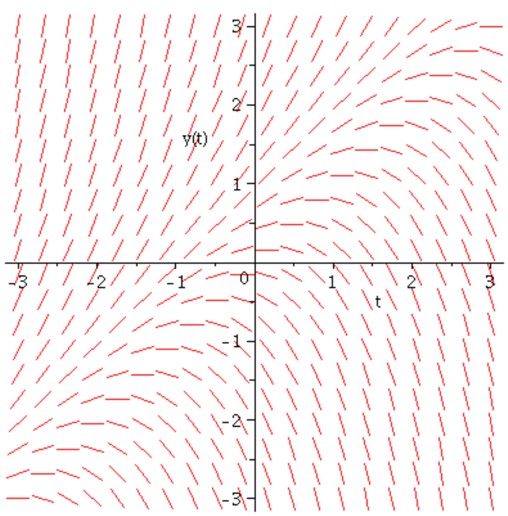
In other graphs, equilibria that are not horizontal lines can also change characteristics as they cross over a zero or a sign change, for instance, two halves of a parabolic-shaped equilibrium curve.

It is usually best to verify your analysis of a complicated direction field by graphing the whole field in the region of interest rather than just test points. For this purpose the direction field grapher at http://www.dartmouth.edu/~rewn/dirfld.html was used for this handout.

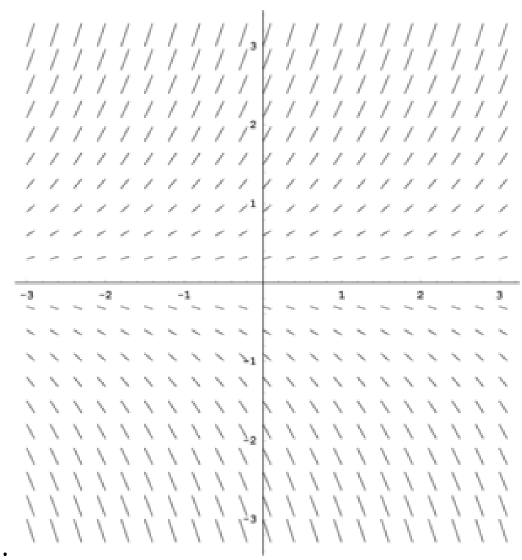
Practice Problems.

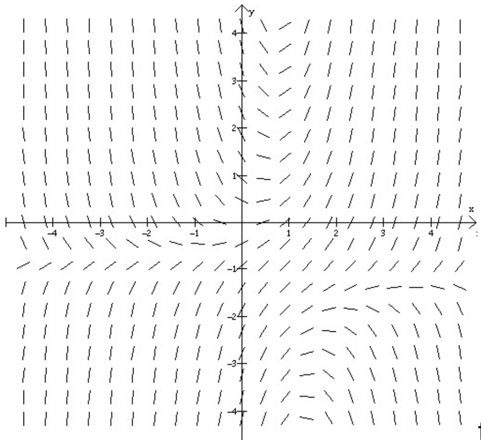
1. Below is a series of direction fields already graphed. Choose three different initial conditions and plot the solution curve going through that point both forwards and backwards in time. Whenever possible, note any equilibria, and state whether they are stable, unstable or semi-stable, or cannot be determined.

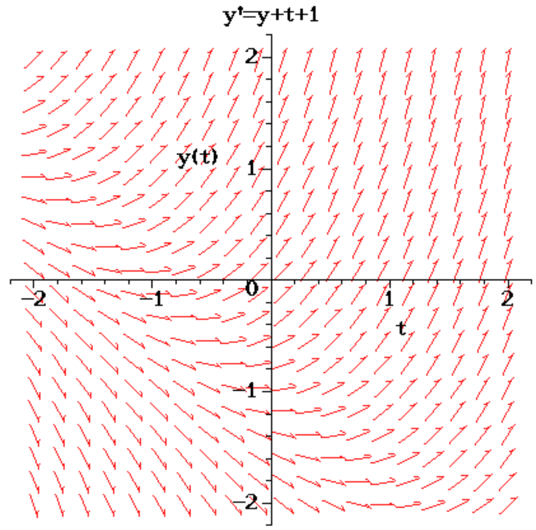
a.

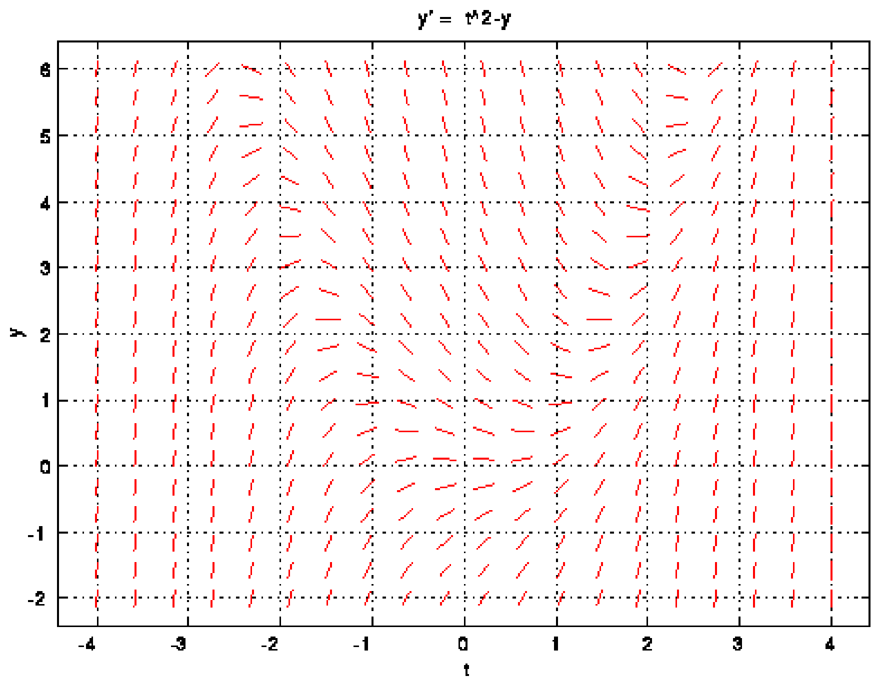


e.

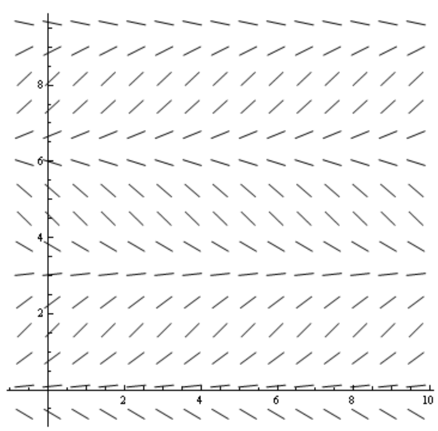


b. 

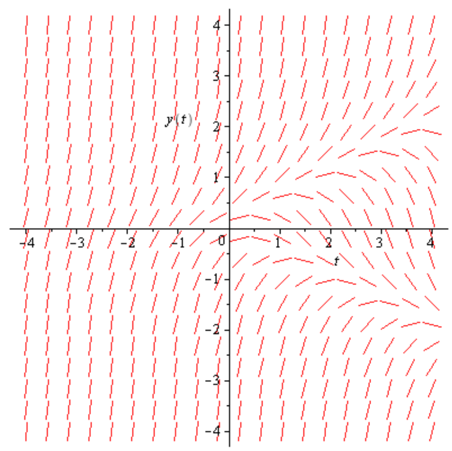
f. 



c.



d.

g. 

2. For each of the following autonomous differential equations, plot the phase plane, and the direction field. On the direction field graph, label the equilibria, and whether the steady state solutions are stable, unstable or semi-stable. Sketch a characteristic solution in each region of the direction field.
   a. $\frac{dy}{dx} = 3 - 2y$
   b. $\frac{dy}{dx} = 1 + 3y$
   c. $\frac{dy}{dx} = y(y-4)$
   d. $\frac{dy}{dx} = -y(5-y)$
   e. $\frac{dy}{dx} = y(y-2)^2$
   f. $\frac{dy}{dx} = y(y-1)(y-2)$
   g. $\frac{dy}{dx} = e^y - 1$
   h. $\frac{dy}{dx} = y - 4\sqrt{y}$
3. For each of the non-autonomous differential equations below, sketch the equilibrium solutions and the slope at key points in each region of the graph. What can you say about the character of the steady state solutions? Are there any vertical slopes on the graph? Verify your insights by graphing the entire field with a computerized direction field grapher (such as the one at http://www.dartmouth.edu/~rewn/dirfld.html). Is there anything interesting you notice about the graph?
   a. $\frac{dy}{dt} = -2 + t - y$
   b. $\frac{dy}{dt} = e^{-t} + y$
   c. $\frac{dy}{dt} = 2t - 1 - y^2$
   d. $\frac{dy}{dt} = 3\sin(t) + 1 + y$
   e. $\frac{dy}{dt} = \frac{1}{3}y^3 - y - \frac{1}{2}t^2$