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Higher-Order Differential Equations:
Mechanical Vibrations & Forcing Functions

We've spent considerable time looking at solution methods for linear ordinary differential equations. We will next look at higher-order equations involving higher-order derivatives. We will begin with second-order equations, equations of the form:

$$a\left(\frac{d^2y}{dt^2}\right)+b\left(\frac{dy}{dt}\right)+cy=0$$

where y is a function of time. Equations of this form can be used to describe electrical circuits and mechanical vibrations, and more.

$$m\left(\frac{d^2y}{dt^2}\right)+p\left(\frac{dy}{dt}\right)+ky=0$$

In the case of mechanical vibrations (of a spring-mass, for instance), m is the mass at the end of the spring, p is the damping term and k is from Hooke's law.

$$L\left(\frac{d^2y}{dt^2}\right)+R\left(\frac{dy}{dt}\right)+\frac{1}{C}y=0$$

The system for a circuit with resistance is similar, where the L is the inductance, R the resistance, and C the capacitance.

$$L\left(\frac{d^2y}{dt^2}\right)+gy=0$$

The solution for a pendulum system (with damping neglected) also belongs to this type of model with L the length of the pendulum and g the gravity constant.

The solution to the homogenous equation, the complementary solution, y_c, is assumed to have the form $y_c(t)=Ae^{mt}$. By applying derivatives $y_c'(t)=Ame^{mt}$ and $y_c''(t)=Am^2e^{mt}$ to the equation above, we find that

$$a\left(Am^2e^{mt}\right)+b\left(Ame^{mt}\right)+cAe^{mt}=0$$

And if we factor out Ae^{mt}, we have the auxiliary equation (sometimes called the characteristic equation) $am^2+bm+c=0$. The zeros of this equation will provide us possible value for m in our complementary solution. Any and all zeros of this function will work, so if there is more than possible zero, the complementary solution will contain both possibilities. Initial conditions will determine the coefficients for each term.

The form of the solutions we obtain for the auxiliary equation will determine the behaviour of the system.

- When the system is undamped (b=0), the system will oscillate forever (c>0)
- When the system is damped (b≠0), three possible situations may occur:
  - The system may be underdamped ($b^2 - 4ac < 0 \Rightarrow$ the roots of the system are complex), the system will oscillate, but the amplitude will approach zero.
  - The system may be critically damped ($b^2 - 4ac = 0 \Rightarrow$ the roots of the system are real and repeating)
  - The system may be overdamped ($b^2 - 4ac > 0 \Rightarrow$ the roots of the system are real and non-repeating). In both of the last two cases, the system will approach zero without undergoing a single oscillation.

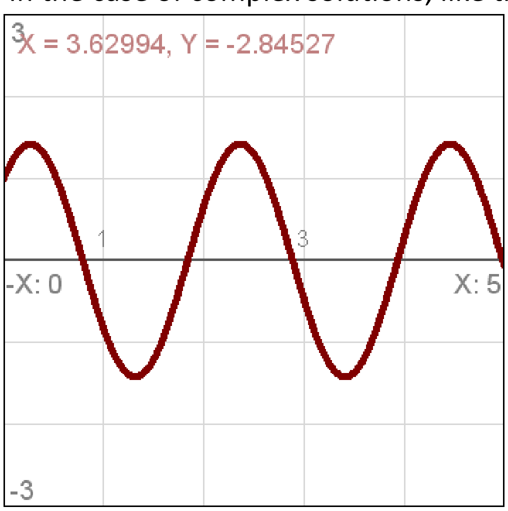
Let's consider some examples.

Example 1: Undamped.

$$\left(\frac{d^2y}{dt^2}\right) + 9y = 0$$

The auxiliary equation is $m^2 + 9 = 0$, giving us m = ±3i.

In the case of complex solutions, like this one, we use the identity (Euler's)



$e^{i\omega t} = \cos\omega t + i\sin\omega t$, so in this case, our solution will be of the form $y_c(t) = c_1\cos(3t) + c_2\sin(3t)$. (The i can be eliminated by choosing an appropriate coefficient, so it is left out of the solution.) The form of this equation makes sense, since when we take two derivatives of sine or cosine functions we end up with constant multiples of the same function. The values of the coefficients must be found using initial conditions.

This graph produced with $c_1 = c_2 = 1$. Changing the amplitude of one trig function produces a phase shift as well as a change in amplitude. Indeed, our result with our coefficients known, can be rewritten as a single sine wave with phase shift. We leave that as an exercise for the reader.

These long-term oscillations are typical of the pendulum example.

Example 2: Underdamped.

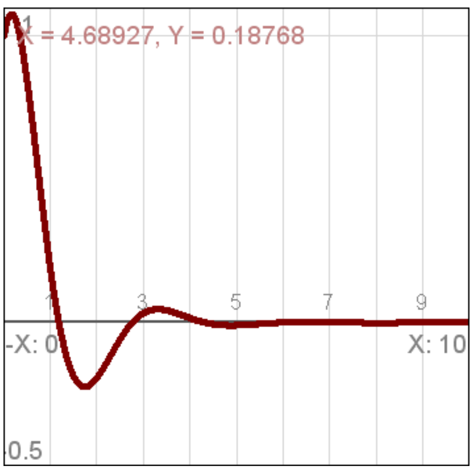
$$\left(\frac{d^2y}{dt^2}\right)+2\left(\frac{dy}{dt}\right)+5y=0$$

Our auxiliary equation is $m^2+2m+5=0$. The roots of this equation must be obtained from the quadratic formula:

$$\frac{-2\pm\sqrt{2^2-4(1)(5)}}{2(1)}=\frac{-2\pm\sqrt{4-20}}{2}=\frac{-2\pm\sqrt{-16}}{2}=-1\pm 2i$$

If we have $e^{mt}=e^{(-1+2i)t}$, applying exponential rules and the identity from Example 1, we get:

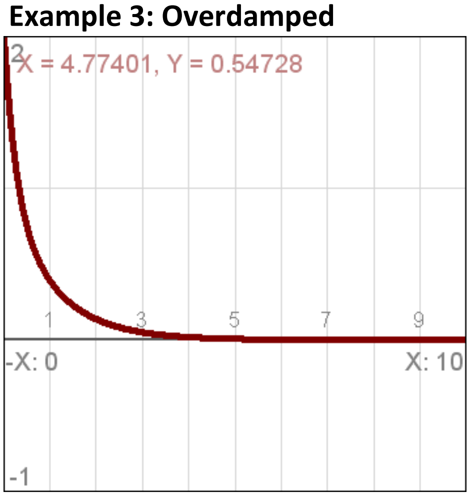
$$e^{mt}=e^{(-1+2i)t}=e^{-t}e^{2it}=e^{-t}(\cos(2t)+i\sin(2t))$$



Thus our solution is of the form $y_c=e^{-t}(c_1\cos(2t)+c_2\sin(2t))$ for our problem. More generally, if our solution is of the form a$\pm$bi, our solutions for the homogeneous case take the form $y_c=e^{at}(c_1\cos(bt)+c_2\sin(bt))$.

As we can see from the graph, the system will oscillate for a time, but with the amplitude decreasing toward zero. For most of the systems we will solve for, the oscillations will appear to die out rather rapidly. Only systems with extremely small damping coefficients will oscillate for any significant length of time on its own.

Example 3: Overdamped

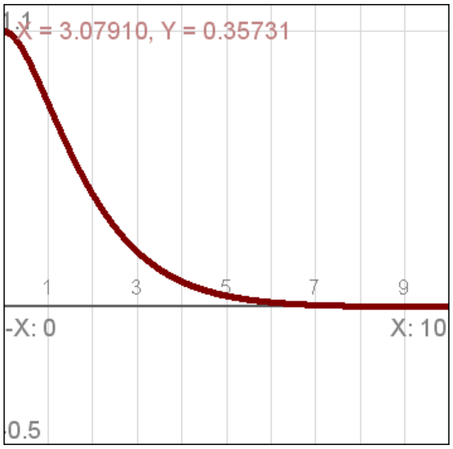


$$\left(\frac{d^2y}{dt^2}\right)+5\left(\frac{dy}{dt}\right)+4y=0$$

In the case of overdamping, the system will not oscillate, not even through a single cycle, but will rather go to zero. The system may pass through zero once before approaching zero, but whether it does or not will depend on the initial conditions of the system.

Our auxiliary equation $m^2+5m+4=0$ has zeros at -1 and -4. Our solution then takes the form $y_c=c_1e^{-t}+c_2e^{-4}$.

Example 4: Critically Damped



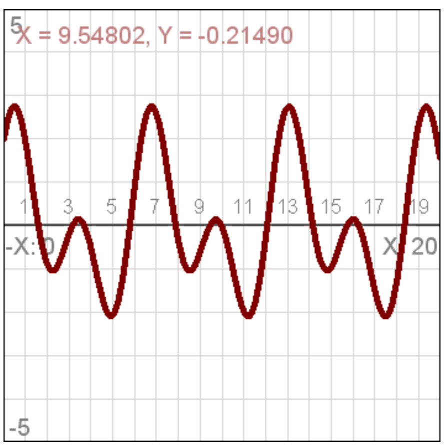
$$\left(\frac{d^2y}{dt^2}\right)+2\left(\frac{dy}{dt}\right)+y=0$$

In the case of a critically damped case, our auxiliary equation results in repeated roots. Here, $m^2+2m+1=0$, and we have a single (repeated) root at m = -1. We will have one component of the solution as e^{-t}, but to capture the influence of the second root, we must find a way for the second term to somehow differentiate itself from the first one. We will therefore multiply the second term by t: te^{-t}.

Thus, $y_c=c_1e^{-t}+c_2te^{-t}$.

Example 5. Higher-Orders

Equations with third and fourth (and higher-order derivatives) can also be solved this way, by finding roots of the auxiliary equation. For example:



$$\left(\frac{d^4y}{dt^4}\right)+5\left(\frac{d^2y}{dt^2}\right)+4y=0$$

This fourth order equation has roots of the auxiliary equation $m^4+5m^2+4=0$ of $\pm i$ and $\pm 2i$. So, our proposed general solution would be $y_c=c_1\sin(t)+c_2\cos(t)+c_3\sin(2t)+c_4\cos(2t)$.

Non-homogeneous Equations

Suppose we consider the homogeneous equation we considered above:

$$a\left(\frac{d^2y}{dt^2}\right)+b\left(\frac{dy}{dt}\right)+cy=0$$

But now, we are going to add a forcing function, i.e.

$$a\left(\frac{d^2y}{dt^2}\right)+b\left(\frac{dy}{dt}\right)+cy=f(t)$$

The forcing function will continue adding energy to the system. If the natural oscillations die out, the forcing function may dominate the resulting behaviour of the system. Or, if the forcing function has the same frequency as the natural frequency of the system, it may cause the system to increase in energy until it fails. Forcing functions are a very important component of modeling real world engineering behaviour. As we will see, the form of the particular solution y_p will depend on the form of the forcing function.

Our final solution for the whole system will be $y(t)=y_c+y_p$

- Real world forcing functions include earthquakes, wind, walking (pedestrians on a bridge, for instance). Can you think of any others?

We can use the following guidelines for determining y_p:

- $f(t)$ is a polynomial: use all terms of degree less than or equal to the degree of the highest term
  - Suppose $f(t)=t^4$, then $y_p=A+Bt+Ct^2+Dt^3+Et^4$
  - Suppose $f(t)=t$, then $y_p=A+Bt$
- $f(t)$ is an exponential: then $y_p=Ae^{kt}$
  - Suppose $f(t)=e^{-2t}$ (and -2 is **not** a root of the auxiliary equation of the homogeneous case), then $y_p=Ae^{-2t}$.
  - Suppose $f(t)=e^{-2t}$ (and -2 **is** a (unrepeated) root of the auxiliary equation of the homogeneous case), then $y_p=Ate^{-2t}$
  - Suppose $f(t)=e^{-2t}$ (and -2 is a (once repeated) root of the auxiliary equation of the homogeneous case), then $y_p=At^2e^{-2t}$.

  Multiply by powers of t until it is higher than any term in y_c.
- $f(t)$ is a sine or cosine function, then $y_p=A\cos\omega t+B\sin\omega t$
  - Suppose $f(t)=\sin(2t)$ (and 2i is **not** a root of the auxiliary equation), then $y_p=A\sin(2t)+B\cos(2t)$
  - Suppose $f(t)=\sin(2t)$ (and 2i **is** a root of the auxiliary equation), then $y_p=At\sin(2t)+Bt\cos(2t)$

  As with the exponentials, multiply by powers of t for repeated roots until your y_p solution has terms which are unique relative to y_c.
- $f(t)$ is a product of terms of these types, we essentially combine the properties. For instance:

- Suppose $f(t) = t^2 \sin t$, then
  $y_p = A\sin t + B\cos t + Ct\sin t + Dt\cos t + Et^2 \sin t + Ft^2 \cos t$

- If $f(t)$ contains multiple terms, each term (or set of related terms) can be taken separately.
  - If $f(t) = x^2 + te^{-2t} + \sin(3t)$ then
    $y_p = A + Bt + t^2 + De^{-2t} + Ete^{-2t} + F\sin(3t) + G\cos(3t)$

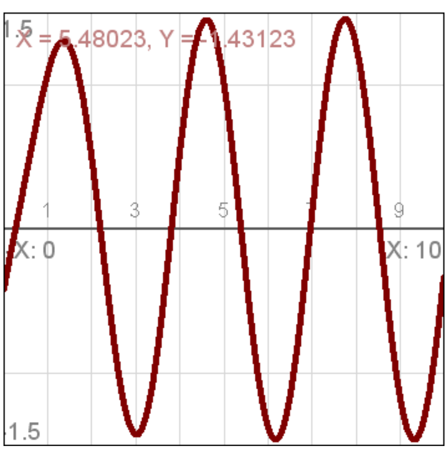
Example 6: Forcing Function on an Underdamped System

$$\left(\frac{d^2y}{dt^2}\right) + 2\left(\frac{dy}{dt}\right) + 5y = 6\sin 2t$$

We found in Example 2 above that $y_c = e^{-t}(c_1\cos(2t) + c_2\sin(2t))$. Even though the frequencies of the sine and cosine functions are the same as our forcing function here, the root is -1±2i, not just ±2i. So, even though they look similar, we don't have to treat this as if we are repeating a root. Thus our y_p has the form $y_p = A\sin(2t) + B\cos(2t)$. While c_1, c_2 depend on initial conditions, the unknown coefficients in y_p depend on the equation. To find them, we'll have to take derivatives and solve from there.

$$y_p = A\sin(2t) + B\cos(2t)$$
$$\frac{dy_p}{dt} = 2A\cos(2t) - 2B\sin(2t)$$
$$\frac{d^2y_p}{dt^2} = -4A\sin(2t) - 4B\cos(2t)$$

$$\left[-4A\sin(2t) - 4B\cos(2t)\right] + 2\left[2A\cos(2t) - 2B\sin(2t)\right] + 5\left[A\sin(2t) + B\cos(2t)\right] = 6\sin(2t)$$
$$-4A\sin(2t) - 4B\cos(2t) + 4A\cos(2t) - 4B\sin(2t) + 5A\sin(2t) + 5B\cos(2t) = 6\sin(2t)$$
$$\left[-4A\sin(2t) - 4B\sin(2t) + 5A\sin(2t)\right] + \left[-4B\cos(2t) + 4A\cos(2t) + 5B\cos(2t)\right] = 6\sin(2t)$$
$$\sin(2t)[A - 4B] + \cos(2t)[B + 4A] = 6\sin(2t)$$
$$A - 4B = 6$$
$$4A + B = 0$$



Solving for A and B, we get $A = \frac{6}{17}, B = -\frac{24}{17}$. Thus $y_p = \frac{6}{17}\sin(2t) - \frac{24}{17}\cos(2t)$ and our complete solution for the system is

$$y(t) = e^{-t}(c_1 \sin(2t) + c_2 \cos(2t)) + \frac{6}{17}\sin(2t) - \frac{24}{17}\cos(2t).$$

- You try it: Let $f(t) = t^2$ for the same differential equation in Example 7. Find y_p.

  Did you get $y_p = -\frac{2}{125} - \frac{4}{25}t + \frac{1}{5}t^2$?

Problems:

For each problem below, solve the homogeneous system.

1. $y'' - 2y' + y = 0$
2. $y'' + 9y = 0$
3. $16y'' - 24y' + 9y = 0$
4. $25y'' - 40y' + 16y = 0$
5. $y''' - 6y'' + 12y' + 8y = 0$ [Hint: if you can't factor, try graphing the auxiliary equation.]
6. $y'' - 2y' + 4y = 0$
7. $y'' + 4y' + 6y = 0$
8. $9y''' + 0.6y'' + 0.01y' = 0$
9. $y'' - 5y' - 14y = 0$
10. $y^{IV} + 2y'' + y = 0$

For each of the problems above, solve for the given forcing functions.

1. $f(t) = e^t, g(t) = e^{-t}$
2. $f(t) = \sin(4t), g(t) = \sin(3t)$
3. $f(t) = e^{-t}, g(t) = te^{-t}$
4. $f(t) = t\sin t, g(t) = t^2$
5. $f(t) = e^{2t}, g(t) = t^2 + te^{2t}$
6. $f(t) = e^{-t}, g(t) = e^{-t}\sin(4t)$
7. $f(t) = t\cos t, g(t) = e^t \cos t$
8. $f(t) = e^{2t}, g(t) = \cos(2t)$
9. $f(t) = t^2 + te^{-t}, g(t) = 6 + \sin t$
10. $f(t) = t\sin t, g(t) = t^2\cos t + e^{-t}$

For #8, set up, but do not solve for $f(t) = t^4 + te^{-t} + t^2\cos t$