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Eigenvalues and Eigenfunctions

Eigenvalues and eigenfunctions (from the German word *eigen* for "inherent" or "characteristic") arise in differential equations largely from boundary value problems (where conditions of the equation are stated at two different points rather than an initial value problem where two conditions are stated at the same point), and where an unknown constant is present. The form of solutions to the equation often depends on the value of the constant. Values that produce only a trivial solution (where y=0 everywhere, for instance) are valid solutions, but not particularly interesting. But sometimes particular values of the constant will produce interesting non-trivial solutions. These values are the eigenvalues, and the solutions they produce are the eigenfunctions that go with them. We will consider several examples of these solutions, starting with the most common type. Solving for eigenvalues and eigenfunctions can be a long process as well as solid command of the algebra involved. How to find the eigenvalues and the conditions on them that need to be tested sometimes depend on the form of the equation and the unknown constant's placement in it. There is, unfortunately, no one method that will work for every possible situation.

Example 1. Find the values of λ that produce non-trivial solutions in the equation $y'' + \lambda y = 0, y(0) = 0, y(L) = 0$. What are the eigenfunctions that go with eigenvalue? For each eigenvalue, is the set of eigenfunctions unique, or infinite?

Start by writing the differential equation as its characteristic equation to solve it.

$$r^2 + \lambda = 0 \rightarrow r^2 = -\lambda$$

This suggests three possible conditions. If λ is negative, -λ will be positive and so the characteristic equation will have two positive real and distinct roots. If λ is positive, the right side of the equation will be negative and produce two purely imaginary roots. If λ is zero, then r is zero and is a repeated root. These three cases will have to be analyzed separately.

Case 1: λ=0 is the simplest case to consider first. This gives us the differential equation $y'' = 0$. Functions whose second derivatives are zero are constants, and linear functions of x. Thus y(x)=Ax+B.

We can look at the boundary conditions now. For y(0)=0, we have 0=A(0)+B, which implies B must be zero. Testing y(π)=0, we have 0=Aπ. But pi can't be zero, so to satisfy this, A must be. This leaves us with the trivial solution y(x)=0 everywhere.

Note: the solutions to these problems depends distinctly on the values of the boundaries. If one of these boundaries was non-zero, one or both of these constants might have survived.

Case 2: Consider the case where λ<0. To ensure that λ is negative, let's define λ=$-\mu^2$. This way, μ can be anything, but λ will always be negative. Making this substitution, our characteristic equation becomes: $r^2 = -(-\mu^2)$ or $r^2 = \mu^2$. This means that $r = \pm\mu$. This gives us the equation $y(x) = Ae^{\mu x} + Be^{-\mu x}$ or $y(x) = Ccosh(\mu x) + Dsinh(\mu x)$. These are equivalent equations for some value of the constants. Which you prefer is up to you. I'm going to use the second version here.

Replacing the initial conditions gives: y(0)=Ccosh(0)+Dsinh(0)=C=0 since cosh(0)=1, and sinh(0)=0. Using the second condition: $y(L) = Dsinh(\mu L) = 0$. The only place sinh(x) is zero is when x is zero. So since sinh(μL)≠0, D must be. That leaves us, as before, with just y(x)=0, the trivial solution.

Case 3: Consider the case where λ>0. To ensure that λ stays positive, set $\lambda = \mu^2$. Now, μ can be anything, but λ will always stay positive. Putting this into our characteristic equation we get $r^2 = -\mu^2$. The solution to this equation is pure imaginary: $r = \pm\mu i$. This results in the solution $y(t) = Acos(\mu t) + Bsin(\mu t)$. Let's look at the initial conditions in this solution: y(0)=Acos(0)+Bsin(0)=0 This implies that A=0. Testing the second initial condition: y(L)=Bsin(μL)=0. This solution will force B to be zero, as in all our other cases only when sinμL≠0. But sometimes sin(μL) will equal zero, and that will happen when μL=nπ, that is to say, any whole number multiple of π. We can solve for μ (and thus for λ), but setting $\mu = \frac{n\pi}{L}$, and $\lambda = \frac{n^2\pi^2}{L^2}$, where L is some given constant and n=±1, ±2, ±3, etc.. For these values of λ, *and only these values of* λ, the solution to the equation will be non-trivial and of the form $y(x) = Bsin(\mu t)$.

The eigenvalues in this case are $\lambda = \frac{n^2\pi^2}{L^2}$ and the eigenfunctions are $y(x) = Bsin(\mu t)$. Because B is undetermined, the eigenfunctions for each eigenvalue are infinite. The set of eigenvalues map onto the set of natural numbers.

Example 2. Find the eigenvalues and eigenfunctions for the boundary value problem $y'' + 4y' + \lambda y = 0, y(0) = 0, y(L) = 0$.

This equation is in quadratic form, so it will help to come up with a condition based on the quadratic formula. The characteristic equation is $r = \frac{-4\pm\sqrt{4^2-4\lambda}}{2} = \frac{-4\pm 2\sqrt{4-\lambda}}{2} = -2 \pm \sqrt{4-\lambda}$.

We can use the quadratic formula solution to come up with our three conditions based on the discriminant: 4-λ. Case 1: 4-λ>0, Case 2: 4-λ=0, Case 3: 4-λ<0.

Case 1. If 4-λ>0, then λ<4. Let's replace 4-λ with μ^2. Then $r = -2 \pm \mu$. These are two real solutions and give a solution to the differential equation as $y(x) = Ae^{(-2+\mu)x} + Be^{(-2-\mu)x} = e^{-2x}(Acosh(\mu x) + Bsinh(\mu x))$. Plugging in our initial conditions, we get y(0)=(1)(Acosh(0)+Bsinh(0))=0 implies that A=0. Using the second set of conditions, $y(L) = e^{-2L}Bsinh(\mu L) = 0$. Since the exponential part is never zero, and the hyperbolic sine is only zero when x is zero, to satisfy this condition, B must equal 0. That leaves us with only the trivial solution, y(x)=0.

Case 2. If 4-λ=0, then λ=4. This leaves us with the repeated root, r= -2. Our solution then is $y(x) = Ae^{-2x} + Bxe^{-2x}$. Test our initial conditions in this equation: y(0)=A+0=0. Thus, A=0. The second condition gives us $y(L) = BLe^{-2L} = 0$. The only way this is possible if B=0. Again, we get the trivial solution y(x)=0.

Case 3. If 4-λ<0, we set $4-\lambda=-\mu^2$. Then $r=-2\pm\mu i$. These complex solutions give $y(x)=e^{-2x}(A\cos(\mu x)+B\sin(\mu x))$. Plug in the initial conditions: y(0)=(1)(Acos(0)+Bsin(0))=0. This implies A=0. The second one gives $y(L)=B\sin(\mu L)=0$. As with the last example, sin(μL)=0 whenever μL=nπ. That gives us $\mu=\frac{n\pi}{L}$, and $4-\lambda=\frac{n^2\pi^2}{L^2}$ or $\lambda=4-\frac{n^2\pi^2}{L^2}$.

The eigenvalues for this equation are $\lambda=4-\frac{n^2\pi^2}{L^2}$ and the eigenfunctions for each are $y(x)=B\sin(\mu x)$ where $\mu^2=4-\lambda$. The eigenfunctions are infinite for each eigenvalue. The set of eigenvalues maps onto the set of natural numbers.

Practice Problems:

For each problem below, find the eigenvalues and eigenfunctions for each problem for any non-trivial solutions. Indicate whether the eigenfunctions for each eigenvalue form a finite or infinite set. Does the set of eigenfunctions map onto a finite set, the natural numbers, or the real numbers?

a. $y''+\lambda y=0, y(0)=0, y'(\pi)=0$
b. $y''+\lambda y=0, y'(0)=0, y'(L)=0$
c. $y''+2\lambda y'+\lambda^2 y=0, y(0)=0, y(L)=1$ [Hint: The characteristic equation for this factors as $(r+\lambda)^2=0$. This reduces to only two cases, |λ|>0, and λ=0.]
d. $y''+\lambda y'+8y=0, y(0)=0, y(L)=0$
e. $y''-\lambda y=0, y(0)=0, y'(L)=1$