# Spring Problems

One of the most important applications in differential equations is the spring problem. Up to four forces are combined to described the motion of the spring. These are: gravity, damping forces, spring forces, and driving forces.

Gravitational forces are given by F=ma or $F = my''$. The second derivative is acceleration.
Damping forces are given by $F = \gamma v$ or $F = \gamma y'$. The first derivative is the velocity.
The spring forces are given by Hooke's law: F=ky.
The driving force, if there is one, must be given in the problem.

The constants in the other three forces may be given in the problem, or derived from information provided in the problem. Which things have to be calculated, and in which units, depend on whether we are working in English units (pounds, slugs, feet, etc.) or whether we are working in SI units (meters, newtons, kilograms, etc.).

Units of force in English units are pounds. Units of force in SI units are newtons.*
Units of mass in English units are slugs. Units of mass in SI units are kilograms.
Units of distance in English units are feet. Units of distance in SI units are meters.
Units of time are seconds in both units.

*Sometimes you will see dynes used instead which are in grams·cm/sec$^2$. In problems containing force in dynes, keep mass in grams, and length in centimeters.

Gravitational acceleration is 32 feet/sec$^2$ in English units, and 9.8 meters/sec$^2$ in SI units. Conversions can be done between mass and force by the relation F=mg, or F/g=m. This is most common in problems in English units since we tend to describe "mass" as weight (pounds) which depends on our current gravity on Earth. Problems will also often give length in inches and centimeters, and these have to be converted to feet and meters so that the units match everywhere in the problem.

One of the most difficult things about spring problems is getting all the coefficients right that go into the equation. Let's look at some examples for each case.

**Mass vs. Weight.**

1. *A mass weighing 2 pounds stretches a spring 6 inches.*
   This "mass" is given in pounds in English units. We have to convert to slugs using F/g=m: $\frac{2\ pounds}{32\ feet/sec^2} = \frac{1}{16}$ slugs =m.
2. *A mass of 100 grams stretches a spring 5 cm.*
   This mass is in grams instead of kilograms. Convert by dividing by 1000: 100 grams = 0.1 kg=m.
3. *A mass of 2 kg is hung from the spring…*
   No conversion needed.
4. *A mass of 1 slug is hung from a spring…*
   No conversion needed.

**Finding the damping constant.**

5. *The mass is attached to a viscous damper with a damping constant of 400 dyn·s/cm.* This damping constant is already given in terms of force/velocity. Here $\gamma$ is given.
6. *If there is no damping…* The damping constant is zero. If they don't say anything about it, it's also assumed to be zero.
7. *A mass… is also attached to a viscous damper that exerts a force of 3 N when the velocity of the mass is 5 m/s.* Here we are given a value for the force (in newtons) and a value for the velocity (in meters/sec). Use the $F = \gamma v$ formula to solve for gamma: $3 = \gamma \cdot 5 \rightarrow \gamma = \frac{3}{5}$.
8. *The motion takes place in a medium that offers a resistance numerically equal to the magnitude of the instantaneous velocity.* The "instantaneous velocity" the just $y'$. Saying that the resistance (force) is "numerically equal" is to say that $\gamma = 1$.
9. *The mass is attached to a dashpot mechanism that has a damping constant of 0.25 lbs·sec/ft.* A dashpot is a damping mechanism. The damping constant is given in the correct units.

**Spring Constant.**

10. *A spring-mass system has a spring constant of 3 N/m.* There is nothing to do here. k=3.
11. *A spring is stretched 6 in by a mass that weighs 8 lbs.* The force in this case is 8 and the distance is 0.5 feet. Putting this into our formula F=ky we have 8=k(0.5) or k=16.
12. *A mass of 5 kg stretches a spring 10 cm.* Here we have a real mass, so to get the force we use F=mg=5*9.8=49. The spring is stretched 10 cm or 0.1 meters. So F=ky gives us 49=k(0.1) or k=490.
13. *A spring is stretched 10 cm by a force of 3 N.* F is given as 3, and the distance is 10 cm = 0.1 meters. So F=ky gives 3=k(0.1) or k=30.
14. *A mass of 20 g stretches a spring 5 cm.* Look at the rest of the problem before deciding on the units. If the problem uses dynes, stay in grams and centimeters for mass and distance, and use 980 cm/sec$^2$ as the gravitational constant to get force in dynes. So F=mg give 20*980=19,600 dynes. And F=ky gives 19,600=k(5) gives k=3920.

**Forcing term.**

15. *If the system is driven by an external force of (3cos(3t)-2sin(3t)) N,…* The external force must be given explicitly in the problem. No calculations should be needed.

**Initial conditions.**

16. *If the mass is pulled down 3 in and then released…* Pulled down 3 in = ¼ feet, but "down" is in the negative y direction, so y(0)= -1/4. "Released" means no initial velocity was imparted, so $y'(0) = 0$.

17. *If the mass is set in motion from its equilibrium position with an initial velocity of 3 cm/s…*
From "equilibrium position" is y(0)=0. And "initial velocity of 3 cm/s" means $y'(0) = 0.03$ m. The problem doesn't say up or down, so you can assume it's the positive direction.

We should now have all the information we need in m, $\gamma$, k, and the forcing term to set up our problem.

When setting up the differential equation for a spring problem, it's sometimes helpful to think of balancing forces. One side of the equation is the force given by the acceleration: ma or $my''$. The other side of the equation has all the other terms. Here, we need to take signs into consideration since the acceleration will be in the opposite direction of the damping force and the spring forces. When we take direction rather than just magnitude into consideration, both get negative signs because the force acts against the motion of the object. So $F = -\gamma y'$ and F=-ky. We ignored these directional negative signs earlier because we were just interested in the magnitude of the constants. These two terms together with the forcing term all go on the right side of the force equation to balance the ma term.

$$my'' = -\gamma y' - ky + f(t)$$

To solve the equation we need to put everything with y or its derivatives on one side, and this is where we get the standard equation we will try to solve, a second order differential equation with constant coefficients.

$$my'' + \gamma y' + ky = f(t)$$

This is also why all the constants on the right are positive even if the equation for the damping and spring constants contain a negative sign.

**Example 1**. A mass weighing 2 lbs. stretches a spring 6 in. If the mass is pulled down an additional 3 in and then released, and if there is no damping, determine the position of u of the mass at any time t. Find the frequency, period and amplitude of the motion.

To find mass, divide 2 by the gravitational constant of 32: 2/32= 1/16 slugs.
The damping constant is 0 since there is no damping.
To find k, 2 lbs is the force, and 6 in = 0.5 feet is the distance: F=ky gives 2=k(0.5) or k=4.
There is no forcing term since it's not mentioned.

Thus our equation becomes: $\frac{1}{16}y'' + 4y = 0$. Solve this equation using the characteristic equation: $\frac{1}{16}r^2 + 4 = 0 \rightarrow r^2 + 64 = 0 \rightarrow r = \pm 8i$. So our solution is y(t)=Acos(8t)+Bsin(8t).

To solve for A and B we need initial conditions. These are stated in the problem:

"If the mass is pulled down an additional three inches…" from its equilibrium position. Assuming that down is in the negative y direction, then "down three inches" translates into -1/4 feet, or y(0)= -1/4. Since the mass is pulled down and merely released, there is no initial velocity, so $y'(0) = 0$.

-1/4=Acos(0)+Bsin(0)=A+0 Thus, A=-1/4.

If y(t)= -1/4cos(8t)+Bsin(8t), then $y'(t) = 2\sin(8t) + 8Bcos(8t)$. Replacing the second condition into the equation gives: 0=2sin(0)+8Bcos(0) or 0=8B, so B=0.

This leaves us with y= -1/4cos(8t).

The **amplitude** is given by the equation $R = \sqrt{A^2 + B^2}$. Since there is only one term, it's just the absolute value of A or ¼.

The **frequency** is just $\omega$ in the expression $Acos(\omega t) + Bsin(\omega t)$. Here, that's 8.

The **period** is given by $T = \frac{2\pi}{\omega} = \frac{2\pi}{8} = \frac{\pi}{4}$.

Where there is damping and we have trig functions together with an exponential, these last two are the **quasi-frequency** and the **quasi-period**, but they are calculated the same way. The quasi-frequency is always smaller than the natural frequency when there is no damping, since the damping will slow the motion. This problem is **undamped**.

**Example 2**. A spring is stretched 6 in by a mass that weighs 8 lbs. The mass is attached to a dashpot mechanism that has a damping constant of 0.25 lbs·sec/ft and is acted on by an external force of 4cos(2t) lbs.

Converting 8 lbs to mass: F/g=8/32 = ¼ slugs.
The damping constant is provided directly.
The spring constant is calculated from F=ky, or 8=k(0.5ft) or k=16.

So our equation becomes: $\frac{1}{4}y'' + \frac{1}{4}y' + 16y = 4\cos(2t)$. To eliminate the fractions, I have to do it with the forcing term included: $y'' + y' + 64y = 16\cos(2t)$.

The characteristic equation for the homogeneous problem is $r^2 + r + 64 = 0$. I'll need the quadratic formula: r=$\frac{-1\pm\sqrt{1^2-4(64)}}{2} = \frac{-1\pm\sqrt{255}i}{2}$. So $r = \lambda \pm \mu i = -\frac{1}{2} \pm \frac{\sqrt{255}}{2}i$.

Getting ugly numbers like this in spring problems is par for the course when damping and real-world problems take place. Given these values for r, we get a solution for y containing both exponentials and trig functions. Solutions contain both these parts are **underdamped**.

$$y_c(t) = e^{-\frac{1}{2}t}\left(Acos\left(\frac{\sqrt{255}}{2}t\right) + Bsin\left(\frac{\sqrt{255}}{2}t\right)\right)$$

Now we need to deal with the forcing term.

The frequency of the homogeneous solution and the forcing term are different, so we assume the solution Y(t)=Ccos(2t)+Dsin(2t) and plug this back into the differential equation along with its derivatives.

$$Y'(t) = -2Csin(2t) + 2Dcos(2t), Y''(t) = -4Ccos(2t) - 4Dsin(2t)$$

$$y'' + y' + 64y =$$
$$-4Ccos(2t) - 4Dsin(2t) - 2Csin(2t) + 2Dcos(2t) + Ccos(2t) + Dsin(2t) = 16\cos(2t)$$

Collecting terms:

$$-4Ccos(2t) + 2Dcos(2t) + Ccos(2t) = 16\cos(2t) \to -3C + 2D = 16$$
$$-4Dsin(2t) - 2Csin(2t) + Dsin(2t) = 0 \to -3D - 2C = 0$$

This leaves us with C=-48/13 and D= 32/13.

$$y(t) = y_c(t) + Y(t) = e^{-\frac{1}{2}t}\left(Acos\left(\frac{\sqrt{255}}{2}t\right) + Bsin\left(\frac{\sqrt{255}}{2}t\right)\right) - \frac{48}{13}\cos(2t) + \frac{32}{13}\sin(2t)$$

Because the exponential exponent is negative, this part of the solution (in red) decays over time and is called the transient solution. The solution from Y(t) (in blue) does not decay because it comes from the driving force. This is the steady state solution and the behaviour of the system will look more and more like only this term is present over time.

This problem did not come with initial conditions in order to solve for A and B. But this is the point in the problem where you would do this if you had them: last, after you have solved for the forcing term, and with both components in your solution. The coefficients will be different if no forcing term is present.

**Practice Problems.**

1. A mass of 100 g stretches a spring 5 cm. If the mass is set in motion from its equilibrium position with a downward velocity of 10 cm/sec, and if there is no damping, determine the position y of the mass at any time t. When does the mass first return to equilibrium? (i.e. when is y=0?)
2. A mass weighing 16 lbs stretches a spring 3 in. The mass is attached to a viscous damper with a damping constant of 2 lbs·sec/ft. If the mass is set in motion from its equilibrium position with a downward velocity of 3 in/sec, find its position y at any time t. Find the time τ such that |y(t)|<0.01 for all t>τ.

3. A mass weighing 8 lbs stretches a spring 1.5 in. The mass is also attached to a viscous damper with coefficient $\gamma$. Determine the value for $\gamma$ for which the spring is critically damped. (A **critically damped** spring is a spring for which the characteristic equation as one repeated real root, i.e. when the discriminant of the quadratic equation is exactly equal to zero. Use that fact to find the condition on $\gamma$. Values of $\gamma$ larger than this will result in an **overdamped** spring, one whose characteristic equation has two distinct real roots.)
4. A mass of 5 kg stretches a spring 10 cm. The mass is acted on by an external force of 10sin(t/2) N and moves in a medium that imparts a viscous force of 2 N when the speed of the mass is 4 cm/sec. If the mass is set in motion from its equilibrium position with an initial velocity of 3 cm/sec, formulate the initial value problem describing the motion of the mass. Solve that equation for y(t).
5. If an undamped spring-mass system with a mass that weighs 6 lbs and a spring constant of 1 lb/in is suddenly set in motion at t=0 by an external force of 4 cos(7t) pounds, determine the positions of the mass at any time and draw a graph of the displacement vs. t. (Be wary of the units in this problem.)
6. A spring-mass system has a spring constant of 3 N/m. A mass of 2 kg is attached to the spring and the motion takes place in a viscous fluid that offers a resistance numerically equal to twice the magnitude of the instantaneous velocity. If the system is driven by an external force of 3cos(3t) N, determine the steady state response.