Differential Equations First Order Homogeneous

1. $x\frac{dy}{dx} = \frac{y-x}{x}\cancel{dx}$ $\quad y' = \frac{y-x}{x}$ $\quad$ let $y = ty$, $x = tx$

$$y' = \frac{ty-tx}{tx} = \frac{\cancel{t}(y-x)}{\cancel{t}x} = \frac{y-x}{x}$$

yes, homogeneous degree 1

let $y = vx$
$y' = v'x + v$

$$v'x + v = \frac{vx-x}{x} \Rightarrow v'x + v = \frac{\cancel{x}(v-1)}{\cancel{x}} \Rightarrow v'x + \cancel{v} = \cancel{v} - 1$$

$$\Rightarrow v'x = -1 \Rightarrow v' = -\frac{1}{x} \Rightarrow \int dv = \int -\frac{1}{x}\,dx \Rightarrow v = -\ln|x| + C =$$

$v = -\ln|Ax|$

$$\boxed{y = -x\ln(Ax)}$$

2. $\frac{(y^2+yx)\cancel{dx}}{x^2} = \cancel{x}\frac{dy}{dx} \Rightarrow \frac{y^2+yx}{x^2} = y'$ $\quad$ let $y = tx$, $x = tx$

$$\Rightarrow \frac{t^2y^2 + ty\,tx}{t^2x^2} = \frac{\cancel{t^2}(y^2+yx)}{\cancel{t^2}x^2} = \frac{y^2+yx}{x^2}$$ degree 2

let $y = vx$
$y' = v'x + v$

$$\frac{v^2x^2 + vx^2}{x^2} = \frac{\cancel{x^2}(v^2+v)}{\cancel{x^2}} = v^2 + \cancel{v} = v'x + \cancel{v} \Rightarrow v^2 = v'x$$

$$\int \frac{dv}{v^2} = \int x\,dx \Rightarrow -\frac{1}{v} = x^2 + C \quad v = -\frac{1}{x^2+C} \quad y = \frac{-1}{x^2+C}\cdot x = \frac{-x}{x^2+C}$$

$$\boxed{y = \frac{-x}{x^2+C}}$$

3. $\frac{dy}{dx} = \frac{x+3y}{3x+y}$ $\quad$ homogeneous, degree 1

let $y = vx$ $\quad y' = v'x + v$

$$v'x + v = \frac{x+3vx}{3x+vx} = \frac{x(1+3v)}{x(3+v)} \Rightarrow v'x + v = \frac{1+3v}{3+v}$$
$-v \qquad -v$

3 cont'd

$v'x = \frac{1-3v-v(3+v)}{3+v} = \frac{1-3v-3v-v^2}{3+v} = \frac{1-6v-v^2}{3+v}$

$\int \frac{3+v}{1-6v-v^2}\,dv = \int \frac{1}{x}\,dx$

$u = 1-6v-v^2$

$\frac{du}{-2} = \frac{(-6-2v)dv}{-2} \Rightarrow -\frac{1}{2}du = (3+v)\,dv$

$-\frac{1}{2}\frac{1}{u}\,du = \int \frac{1}{x}\,dx$

$-\frac{1}{2}\ln u = \ln x \Rightarrow \ln\frac{1}{\sqrt{u}} = \ln x + C \Rightarrow \frac{1}{\sqrt{u}} = Ax \Rightarrow \sqrt{u} = \frac{1}{Ax}$

$\Rightarrow u = \frac{1}{A^2x^2} \Rightarrow 1-6v-v^2 = \frac{1}{A^2x^2} \Rightarrow v^2+6v\cancel{-1} = -\frac{1}{A^2x^2}+1$

$v^2+6v+9 = -\frac{1}{A^2x^2}+10 \Rightarrow (v+3)^2 = -\frac{1}{A^2x^2}+10 \Rightarrow$

$v+3 = \pm\sqrt{10-\frac{1}{A^2x^2}} \Rightarrow v = -3 \pm \sqrt{10-\frac{1}{A^2x^2}}$

$\boxed{y = -3x \pm x\sqrt{10-\frac{1}{A^2x^2}}}$

4. $\frac{y\,dx}{x+\sqrt{xy}} = \cancel{(x+\sqrt{xy})}\frac{dy}{dx} \Rightarrow \frac{y}{x+\sqrt{xy}} = y'$ homogeneous degree 1

let $y = vx \quad y' = v'x + v$

$\frac{vx}{x+\sqrt{x^2v}} = \frac{vx}{x+x\sqrt{v}} = \frac{\cancel{x}}{\cancel{x}(1+\sqrt{v})} = \frac{1}{1+\sqrt{v}} = v'x+v \Rightarrow$

$\frac{1}{1+\sqrt{v}} - v\frac{(1+\sqrt{v})}{1+\sqrt{v}} = v'x \Rightarrow \frac{1-v-v^{3/2}}{1+\sqrt{v}} = v'x \Rightarrow$

$\int \frac{1+\sqrt{v}\,dv}{1-v-v^{3/2}} = \int \frac{1}{x}dx \qquad u = 1-v-v^{3/2} \quad -du = +1+\frac{3}{2}v^{1/2}\,dv$

you'll need a computer algebra system to complete this integration

$\int \frac{1+\frac{3}{2}v^{1/2}}{1-v-v^{3/2}}\,dv - \int \frac{\frac{1}{2}v^{1/2}}{1-v-v^{3/2}}\,dv = \ln x$

doable by subst

needs computer help.

5. $y' = \dfrac{y+\sqrt{x^2-y^2}}{x}$ homogeneous degree 1

let $y = vx$ $\quad y' = v'x + v$

$$v'x + v = \frac{vx + \sqrt{x^2 - v^2x^2}}{x} = \frac{vx + x\sqrt{1-v^2}}{x} = \frac{\cancel{x}(v+\sqrt{1-v^2})}{\cancel{x}} \quad -v$$

$-v$

$$v'x = \sqrt{1-v^2} \Rightarrow \int \frac{dv}{\sqrt{1-v^2}} = \int \frac{1}{x}\,dx \qquad \arcsin v = \ln x + c = \ln(Ax)$$

$$v = \sin(\ln x)$$

$$\boxed{y = x\sin(\ln Ax)}$$

6. $\dfrac{x + ye^{y/x}\,\cancel{dx}}{xe^{y/x}} = \cancel{xe^{y/x}}\dfrac{dy}{dx} \Rightarrow y' = \dfrac{x+ye^{y/x}}{xe^{y/x}}$ homogeneous degree 1

let $y = vx$ $\quad y' = v'x + v$

$$v'x + v = \frac{x + vxe^{vx/x}}{xe^{vx/x}} = \frac{\cancel{x}(1+ve^v)}{\cancel{x}e^v} = \frac{1+ve^v}{e^v} \quad -v$$

$-v$

$$v'x = \frac{1 + \cancel{ve^v} - \cancel{ve^v}}{e^v} = \frac{1}{e^v} \Rightarrow \int e^v\,dv = \int \frac{1}{x}\,dx$$

$$e^v = \ln x + C = \ln(Ax) \Rightarrow v = \ln(\ln(Ax))$$

$$y = x\ln(\ln(Ax)) \qquad 0 = 1\cdot\ln(\ln(A)) \Rightarrow e^0 = \ln A \Rightarrow 1 = \ln A$$

$$\Rightarrow A = e$$

$$y = x\ln(\ln ex)$$

7. $\dfrac{y\,\cancel{dx} = \cancel{x(1+\ln y - \ln x)}\dfrac{dy}{dx}}{x(1+\ln y - \ln x)} \Rightarrow \dfrac{y}{x(1-\ln(y/x))} = y'$ homogeneous degree 1

let $y = vx \quad y' = v'x + v$

$$v'x + v = \frac{v\cancel{x}}{\cancel{x}(1 - \ln(\frac{v\cancel{x}}{\cancel{x}}))} = \frac{v}{1 - \ln v}$$

$$v'x = \frac{v}{1-\ln v} - v\left(\frac{1-\ln v}{1-\ln v}\right) = \frac{\cancel{v} - \cancel{v} - \ln v}{1 - \ln v}$$

$$\int dv \frac{(1-\ln v)}{-\ln v} = \int \frac{1}{x}\,dx \Rightarrow \int\left(\frac{-1}{\ln v} + 1\right)dv = \ln x + C = \ln(Ax)$$

needs a computer algebra system to integrate

8. $y' = \frac{x+y}{2x}$ homogeneous of degree 1

$y = vx \quad y' = v'x + v$

$$v'x + \cancel{v} = \frac{x + vx}{2x} = \frac{\cancel{x}(1+v)}{2\cancel{x}} = \frac{1+v}{2} - \frac{2v}{\cancel{2}}$$

$$v'x = \frac{1-v}{2}$$

$$\frac{2}{1-v}\,dv = \frac{1}{x}\,dx \Rightarrow 2\ln|1-v| = \ln x + C$$

$$\cancel{\ln}(1-v)^2 = \cancel{\ln} Ax$$

$$(1-v)^{\cancel{2}} = \pm\sqrt{Ax}$$

$$1 - \sqrt{Ax} = v \qquad y = x(1 \pm \sqrt{Ax})$$

9. $y' = \frac{xy}{x^2 - y^2}$ homogeneous degree 2

$y = vx \quad y' = v'x + v$

$$v'x + \cancel{v} = \frac{vx^2}{x^2 - v^2x^2} = \frac{\cancel{x^2}v}{\cancel{x^2}(1-v^2)} - \frac{v(1-v^2)}{1-v^2} = \frac{\cancel{v} - \cancel{v} + v^3}{1-v^2}$$

$$\frac{1-v^2}{v^3}\,dv = \frac{1}{x}\,dx \Rightarrow \int \frac{1}{v^3} - \frac{1}{v}\,dv = \int \frac{1}{x}\,dx$$

$$-\tfrac{1}{2}v^{-2} - \ln v = \ln x + C \qquad \text{cannot solve explicitly for } v$$

replace $v = y/x \qquad -\frac{1}{2}\frac{x^2}{y^2} - \ln(y/x) = \ln Ax$

10. $y' = \dfrac{2x+3y}{x}$ homogeneous degree 1

$y = vx,\ y' = v'x + v$

$v'x + \cancel{v} = \dfrac{2x+3vx}{x} = \dfrac{\cancel{x}(2+3v)}{\cancel{x}} - v = 2+2v = 2(1+v)$

$v'x = 2(1+v) \Rightarrow \displaystyle\int \frac{1}{1+v}\,dv = \int \frac{2}{x}\,dx \Rightarrow \ln|1+v| = 2\ln x + C$

$\cancel{\ln}|1+v| = \cancel{\ln}|Ax^2| \Rightarrow 1+v = Ax^2 \qquad v = Ax^2 - 1$

$$\boxed{y = x(Ax^2 - 1)}$$

11. $(x\sec(y/x) + y)\,dx - x\,dy = 0 \qquad y(1) = 0$

let $y = ty$, $x = tx$

$\left(tx\sec\left(\frac{\cancel{t}y}{\cancel{t}x}\right) + ty\right)dx - tx\,dy = 0$

$t\left[(x\sec(y/x) + y)\,dx - x\,dy\right] = 0$ homogeneous degree 1

$y = vx,\ y' = v'x + v$

$\dfrac{(x\sec(y/x) + y)\cancel{dx}}{x} = \cancel{x}\dfrac{dy}{dx} \Rightarrow v'x + \cancel{v} = \dfrac{x\sec(\frac{v\cancel{x}}{\cancel{x}}) + vx}{x} = \dfrac{\cancel{x}(\sec v + v)}{\cancel{x}} - v$

$v'x = \sec v \Rightarrow \displaystyle\int \cos v\,dv = \int \frac{1}{x}\,dx \Rightarrow \sin v = \ln x + C = \ln(Ax)$

$v = \arcsin(\ln Ax) \qquad y = x\arcsin(\ln(Ax))$

$0 = 1\arcsin(\ln A(1)) \Rightarrow \arcsin(\ln A) = 0 \Rightarrow \sin 0 = \ln A$

$\Rightarrow 0 = \ln A \Rightarrow A = 1$

$$\boxed{y = x\arcsin(\ln(x))}$$