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Systems of Linear ODEs

Systems of ordinary differential equations can be solved in much the same way as discrete dynamical systems if the differential equations are linear. We will focus here on the case with two variables only, but these procedures can be extended to higher order linear systems (with more variables, or by means of a substitution, derivatives of a higher order). This handout will focus primarily on solving the linear system, the form of solutions, and the behaviour of the origin.

I. Solving a linear ordinary differential equation in one variable.

A linear ODE (ordinary differential equation) in one variable has the general form $y' = ky$. To solve this system, we can perform some algebra to obtain a solution. The technique is called separation of variables: first isolate all the y terms on one side, and any terms without y on the other. Then integrate both sides. To do this algebra, we usually write $y' = \frac{dy}{dt}$ to make the operation more explicit.

$$\frac{dy}{dt} = ky$$

Divide by y and multiply by dt gives:

$$\frac{dy}{y} = kdt$$

Then integrate both sides with respect to the appropriate variable.

$$\int \frac{1}{y} dy = \int k\, dt$$

$$\ln|y| = kt + C$$

$$y = e^{kt+C}$$

A little bit of algebra can rearrange this into a more familiar form.

$$y = e^{kt}e^{C}$$

Let $e^{C} = A$ to obtain:

$$y = Ae^{kt}$$

Where A is some unknown constant to be determined by any initial conditions that might be provided. And we can do a quit test to show that this does satisfy the original differential equation.

$y' = Ae^{kt} \cdot k$ by the chain rule, and since $y = Ae^{kt}$, this can be rewritten as $y' = ky$.

When we go to systems of linear ODEs rather than just the one, it would be great if we could obtain a solution to the equation $\vec{x}' = A\vec{x}$, with A being some $n \times n$ matrix, by making a similar assumption about the form of the solution, namely that it's something like of $\vec{x} = e^{A}\vec{c}$, with $\vec{c}$ a constant vector to be determined by initial conditions. We can't obtain the solution by integrating as we did before because we can't divide by vectors or matrices, and it turns out this form of the solution does work, but before we do that: what the heck is e^{A} anyway?

II. Raising e to a matrix power.

To define e^{A}, the matrix A must be defined for all whole number powers of A, which is to say, it must be square.

Since $e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, we can similarly define

$$e^{A} = \sum_{n=0}^{\infty} \frac{A^n}{n!} = I + A + \frac{A^2}{2} + \frac{A^3}{6} + \frac{A^4}{24} + \cdots$$

This expression is defined for all powers of A, with $A^0 = I$.

If A is some general matrix, this can be very hard to compute by hand, but if A is diagonalizable, it becomes much easier. Suppose A is already diagonal, then in the 2×2 case, if $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$, then $A^n = \begin{bmatrix} a^n & 0 \\ 0 & d^n \end{bmatrix}$, and so

$$e^{A} = I + A + \frac{A^2}{2} + \frac{A^3}{6} + \frac{A^4}{24} + \cdots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} + \frac{1}{2}\begin{bmatrix} a^2 & 0 \\ 0 & d^2 \end{bmatrix} + \frac{1}{6}\begin{bmatrix} a^3 & 0 \\ 0 & d^3 \end{bmatrix} + \frac{1}{24}\begin{bmatrix} a^4 & 0 \\ 0 & d^4 \end{bmatrix} + \cdots$$

Adding corresponding entries gives us

$$\begin{bmatrix} 1 + a + \frac{a^2}{2} + \frac{a^3}{6} + \frac{a^4}{24} + \cdots & 0 \\ 0 & 1 + d + \frac{d^2}{2} + \frac{d^3}{6} + \frac{d^4}{24} + \cdots \end{bmatrix} = \begin{bmatrix} \sum_{n=0}^{\infty} \frac{a^n}{n!} & 0 \\ 0 & \sum_{n=0}^{\infty} \frac{d^n}{n!} \end{bmatrix} = \begin{bmatrix} e^a & 0 \\ 0 & e^d \end{bmatrix}$$

So, it just raises the diagonal entries to be powers of e.

If the matrix A is not diagonal, but can be diagonalized, i.e. $A = PDP^{-1}$, then we can find the matrix $e^{A} = Pe^{D}P^{-1}$, which follows from some simple matrix properties.

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In the examples we'll be working with, all the matrices will be diagonalizable on the set of Complex numbers.

It's because of this property of diagonalization that we will need to find the eigenvalues and eigenvectors of the A matrix that defines our system of differential equations.

III. Solving for the general solution for a system of linear ODEs.

Consider the set of linear ODEs in two variables:

$$\frac{dx_1}{dt} = 3x_1 - 2x_2$$
$$\frac{dx_2}{dt} = 2x_1 - 2x_2$$

If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ we can rewrite this system as

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

or

$$\vec{x}' = A\vec{x}, with\ A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$$

This system conforms to the type of problem we wish to solve. And so the first step in the process will be to find the eigenvalues and eigenvectors needed to diagonalize the matrix. If $\lambda_1\ and\ \lambda_2$ are the eigenvalues of the matrix, then $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, and $e^D = \begin{bmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{bmatrix}$. The similarity transformation $P = [\vec{v}_1 \quad \vec{v}_2]$, where $\vec{v}_1$ is the eigenvector corresponding to the eigenvalue λ_1, and $\vec{v}_2$ is the eigenvector corresponding to λ_2. If we apply the transformation to e^D, we get $\vec{x} = e^{At}\vec{c} = Pe^{Dt}P^{-1}\vec{c}$ and so if we write the solution in vector form, we obtain the following vector form of the solution.

$$\vec{x} = c_1\vec{v}_1e^{\lambda_1 t} + c_2\vec{v}_2e^{\lambda_2 t}$$

To obtain this solution, begin with finding the eigenvalues for the matrix.

$$A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}, A - \lambda I = \begin{bmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{bmatrix}$$

The characteristic equation is $(3-\lambda)(-2-\lambda) + 4 = 0$ or $\lambda^2 - \lambda - 2 = 0$. This factors as $(\lambda - 2)(\lambda + 1) = 0$. Therefore, $\lambda_1 = 2, \lambda_2 = -1$.

Next, we find each eigenvector.

For $\lambda_1 = 2, A - \lambda_1 I = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix}$. This system is dependent, and we find the required eigenvector satisfies the system

$$\begin{matrix} x_1 - 2x_2 = 0 \\ x_2 = (free) \end{matrix} \quad or \quad \begin{matrix} x_1 = 2x_2 \\ x_2 = x_2 \end{matrix}$$

Thus, $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

For $\lambda_2 = -1, A - \lambda_2 I = \begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix}$. This system is also dependent, and the required eigenvector satisfies the system

$$\begin{matrix} 2x_1 - x_2 = 0 \\ x_2 = (free) \end{matrix} \quad or \quad \begin{matrix} x_1 = \frac{1}{2}x_2 \\ x_2 = x_2 \end{matrix}$$

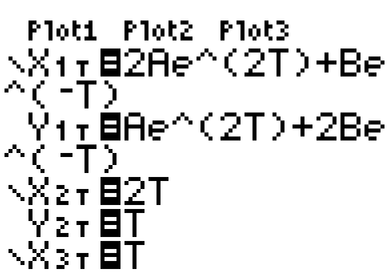
Thus, $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

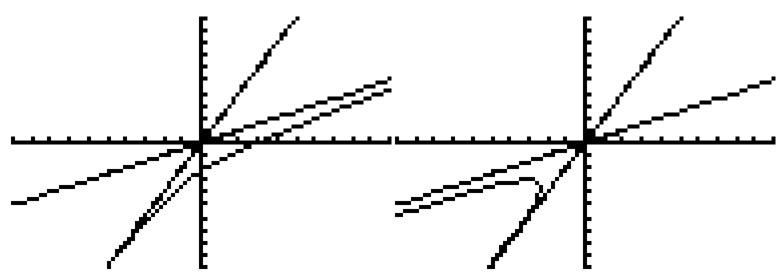
Thus, the solution to our system is $\vec{x} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t}$. We could write these as two separate equations if we wanted to plot them in our calculators (using the parametric functions screen).

$$\begin{cases} x_1 = 2c_1 e^{2t} + c_2 e^{-t} \\ x_2 = c_1 e^{2t} + 2c_2 e^{-t} \end{cases}$$

You can graph the real eigenvectors by remember the relationship of a vector to the slope of a line. If a vector is $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$, then the slope of the line passing through the origin is $m = \frac{b}{a} = \frac{\Delta y}{\Delta x}$. And thus the equation of the line is $y = \frac{b}{a}x$, or in parametric form $x = at, y = bt$. Using this, I can graph the vectors along with the solutions.

Enter the equations on the parametric functions screen. I've used A and B to stand in for the values of c_1, c_2 so that they can be adjusted without resetting the equation. Using the values $A = 2, B = -1$ produces the first graph, and $A = -1, B = -1$ produces the second.





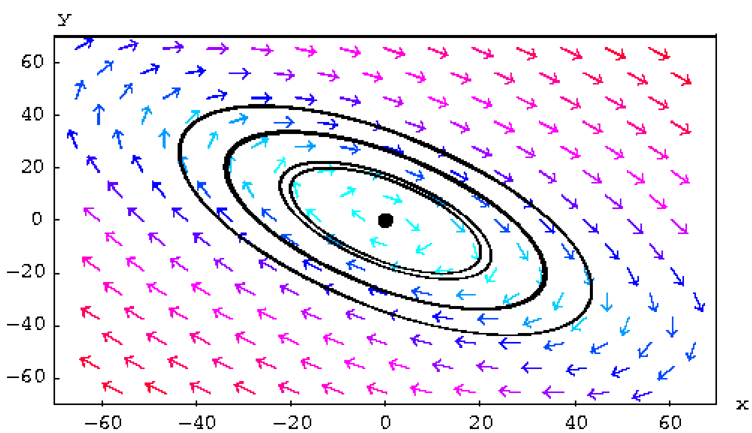
The constants are determined by initial conditions, such as $\vec{x}(0) = \begin{bmatrix} 10 \\ 12 \end{bmatrix}$. If we wanted to plot a variety of initial conditions, we can test values of c_1, c_2 essentially randomly to see what happened. However, like with discrete dynamical systems, the trajectories are going to behave in predictable ways based on the eigenvalues.

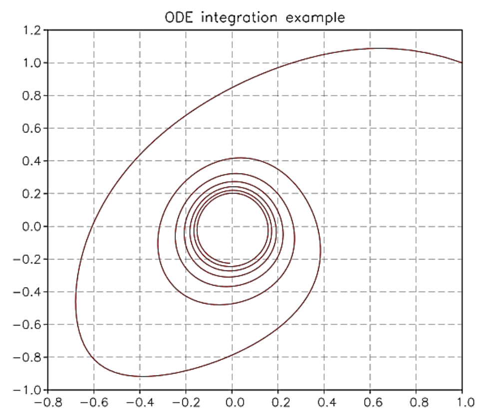
Since raising e to a positive value ($a > 0$), then $e^a > 1$, and so, as t increases, so does e^{at}. Since raising e to a negative value ($b < 0$), then $e^b < 1$, and so as t increases, e^{bt} goes to zero. Both the eigenvalues in our example are real, but it turns out that only the real part of the eigenvalues matter to determine the behaviour of the origin.

If all $Re(\lambda_i) > 0$, then the origin is a repeller. If all $Re(\lambda_i) < 0$, then the origin is an attractor. If some $Re(\lambda_i) > 0$, and some are $Re(\lambda_i) < 0$, then the origin is a saddle point. If any of the real parts of zero, then the origin is stable in that direction (like $\lambda = 1$ for the discrete case).

The complex case, as in the discrete case, induces a rotation, and like the discrete case, since the real parts are the same for both eigenvalues, the origin either attracts or repels, and cannot be a saddle point. Thus we have the same range of trajectory patterns as we saw for the discrete systems, and the graphs of the general behaviour can be obtained in the same way.

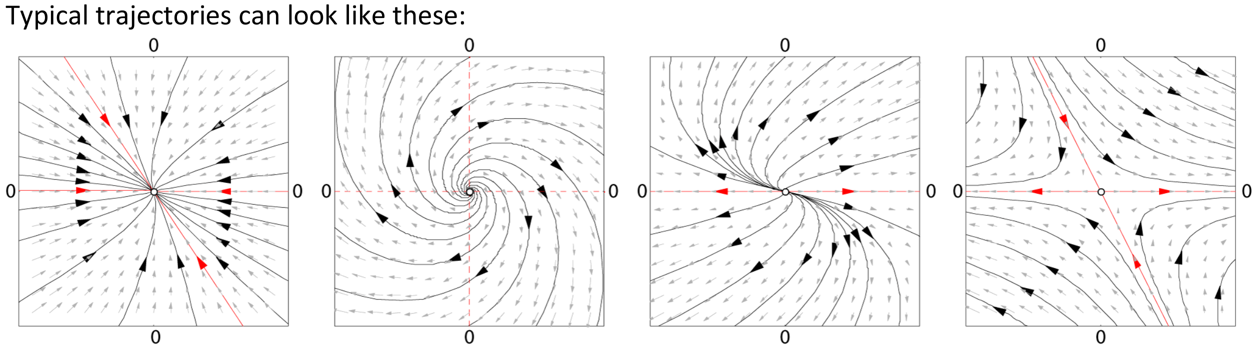
Complex eigenvalues can produce a graph like these:





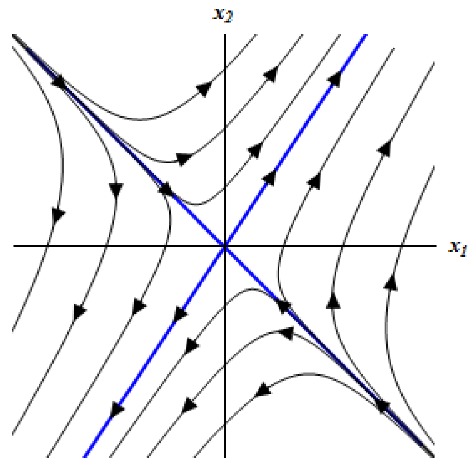
The first graph is a stable complex system (pure imaginary eigenvalues), while the second is spiraling into or away from the origin. (Draw labels on your trajectories to mark the direction.) There are no eigenvectors graphed because the two eigenvectors are complex.

Typical trajectories can look like these:



The first graph is an attractor (all real eigenvalues, both negative). The second is a repeller with complex eigenvalues (real parts are positive). The third graph is a repeller (all real eigenvalues, both positive). The fourth graph is a saddle point (real eigenvalues, one positive and one negative). The eigenvectors are marked in red (on the colour version).

Another example with the trajectories and eigenvectors plotted more clearly:



IV. Complex Case

The complex case follows many of the same steps, but more work with the complex numbers is required and so these problems take a bit more algebra. Let's go through one example.

Consider the system $\vec{x}' = \begin{bmatrix} -7 & 10 \\ -4 & 5 \end{bmatrix} \vec{x}$.

The characteristic equation is $(-7-\lambda)(5-\lambda)+40 = \lambda^2 + 2\lambda + 5 = 0$. This does not factor, and so we use the quadratic formula.

$$\lambda = \frac{-2 \pm \sqrt{4-4(5)}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

We know already that since the real part is negative, the origin with attract, and the system will spiral in because the eigenvalues are complex.

Next, we need the eigenvectors.

$$A - \lambda_1 I = \begin{bmatrix} -6-2i & 10 \\ -4 & 6-2i \end{bmatrix}$$

This matrix is dependent and so our vector must satisfy

$$\begin{matrix} -4x_1 + (6-2i)x_2 = 0 \\ x_2 = free \end{matrix} \qquad or \qquad \begin{matrix} x_1 = \dfrac{3-i}{2} x_2 \\ x_2 = x_2 \end{matrix}$$

Thus the eigenvector is $\vec{v}_1 = \begin{bmatrix} 3-i \\ 2 \end{bmatrix}$. Since the vectors are conjugates just like the eigenvalues, the second vector is $\vec{v}_2 = \begin{bmatrix} 3+i \\ 2 \end{bmatrix}$. However, it turns out there is trick so that we will only need to use the one.

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If we follow the form of the original solution we need a form like this: $\vec{x} = c_1\vec{v}_1e^{\lambda_1 t} + c_2\vec{v}_2e^{\lambda_2 t}$, however, since both our vectors and exponential parts are partially really and partially complex, and that they are conjugates of each other, we can just work with one of the forms, split the result into fully really and fully complex forms, and use just those since the c's can be choose to eliminate only the real or only the complex solutions if we choose the appropriate complex values. This technique allows us to avoid that step.

Consider the solution for λ_1. $\vec{x} = \begin{bmatrix} 3-i \\ 2 \end{bmatrix} e^{(-1+2i)t}$. Recall from exponent properties we can rewrite $e^{(-1+2i)t} = e^{-t}e^{2it}$, and Euler's theorem allows us to write $e^{\theta i} = \cos(\theta) + i\sin(\theta)$, so $e^{(-1+2i)t} = e^{-t}[\cos(2t) + i\sin(2t)]$. Then, to separate the real and imaginary parts, we must distribute with the vector.

$$\begin{bmatrix} 3-i \\ 2 \end{bmatrix} e^{(-1+2i)t} = \begin{bmatrix} 3-i \\ 2 \end{bmatrix} e^{-t}[\cos(2t) + i\sin(2t)] =$$

$$e^{-t}\begin{bmatrix} 3\cos(2t) + 3i\sin(2t) - i\cos(2t) + \sin(2t) \\ 2\cos(2t) + 2i\sin(2t) \end{bmatrix}$$

Now separate the terms with i and those without.

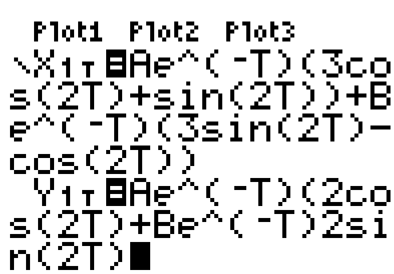
$$e^{-t}\begin{bmatrix} 3\cos(2t) + \sin(2t) \\ 2\cos(2t) \end{bmatrix} + e^{-t}\begin{bmatrix} 3\sin(2t) - \cos(2t) \\ 2\sin(2t) \end{bmatrix} i$$

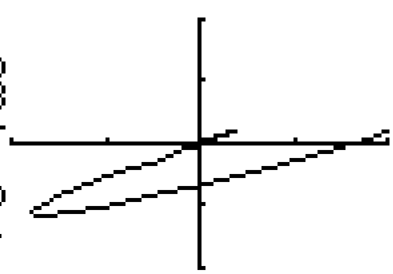
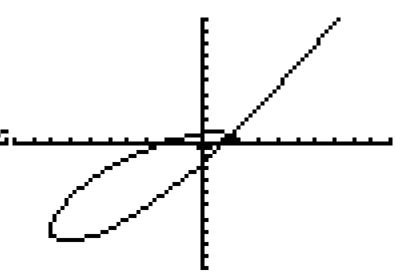
Our solution to the system them has the form

$$\vec{x} = c_1e^{-t}\begin{bmatrix} 3\cos(2t) + \sin(2t) \\ 2\cos(2t) \end{bmatrix} + c_2e^{-t}\begin{bmatrix} 3\sin(2t) - \cos(2t) \\ 2\sin(2t) \end{bmatrix}$$

If you wish to plot these on a graph, do it parametrically so some selected values of c_1, c_2, with the top part of the vector being x, and the bottom part being y.

As with the first example, I used A and B for c_1, c_2 so that the equations can be changed easily. With $A = 2, B = -1$ we get the first graph. The second graphs has $A = 10, B = 20$. Both graphs spiral in as expected.



V. Practice Problems.

For each of the problems below (1-9), find the general solution to the system of linear ODEs, and plot a few trajectories of the system. Be sure to use arrows on the trajectories and eigenvectors to indicate the direction of motion.

1. $\vec{x}' = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \vec{x}$

2. $\vec{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \vec{x}$

3. $\vec{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \vec{x}$

4. $\vec{x}' = \begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix} \vec{x}$

5. $\vec{x}' = \begin{bmatrix} 7 & -1 \\ 3 & 3 \end{bmatrix} \vec{x}$

6. $\vec{x}' = \begin{bmatrix} 4 & -3 \\ 6 & -2 \end{bmatrix} \vec{x}$

7. $\vec{x}' = \begin{bmatrix} -2 & 1 \\ -8 & 2 \end{bmatrix} \vec{x}$

8. $\vec{x}' = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \vec{x}$

9. $\vec{x}' = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix} \vec{x}$

For each of the three dimensional problems below (10-12), find the general solution to the system. Describe the character of the origin. [Hint: if the real part of all the eigenvalues are positive, it's a repeller; if real part of all the eigenvalues are negative, it's an attractor; if there is any sign change in the real part, then the origin is a saddle point.]

10. $\vec{x}' = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} \vec{x}$

11. $\vec{x}' = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix} \vec{x}$

12. $\vec{x}' = \begin{bmatrix} -8 & -12 & -6 \\ 2 & 1 & 2 \\ 7 & 12 & 5 \end{bmatrix} \vec{x}$