Differential Equations Undetermined Coefficients Key

1. $y''-2y'-3y=3e^{2t}$ $\qquad y_1=e^{3t},\ y_2=e^{-t}$

$y''-2y'-3y=0$ $\qquad y_c(t)=c_1e^{3t}+c_2e^{-t}$

$r^2-2r-3=0$

$(r-3)(r+1)=0$

$r=3,\ r=-1$

$Y(t)=Ae^{2t} \qquad Y'(t)=2Ae^{2t} \qquad Y''(t)=4Ae^{2t}$

$4Ae^{2t}-4Ae^{2t}-3Ae^{2t}=3Ae^{2t} \Rightarrow A=-1$

$Y_p(t)=-e^{2t}$

$y(t)=c_1e^{3t}+c_2e^{-t}-e^{2t}$

2. $y''-2y'-3y=3\sin t$ $\qquad y_c(t)=c_1e^{3t}+c_2e^{-t}$ (see above)

$Y(t)=A\cos t+B\sin t \qquad Y'(t)=-A\sin t+B\cos t$

$Y''(t)=-A\cos t-B\sin t$

$-A\cos t-B\sin t-2(-A\sin t+B\cos t)-3(A\cos t+B\sin t)=3\sin t$

$(-A-2B-3A)\cos t+(-B+2A-3B)\sin t=3\sin t$

$-4A-2B=0 \qquad (2A-4B=3)2$

$4A-8B=6$

$-10B=6 \qquad 2A-4(-\frac{3}{5})=3$

$B=-\frac{6}{10}=-\frac{3}{5} \qquad 2A+\frac{12}{5}=3 \Rightarrow 2A=\frac{15}{5}-\frac{12}{5}=\frac{3}{5} \Rightarrow A=\frac{3}{10}$

$Y_p(t)=\frac{3}{10}\cos t-\frac{3}{5}\sin t$

$y(t)=c_1e^{3t}+c_2e^{-t}+\frac{3}{10}\cos t-\frac{3}{5}\sin t$

3. $y''-2y'-3y=3e^{2t}\cos t$ $\qquad y_c(t)=c_1e^{3t}+c_2e^{-t}$ (see above)

$Y(t)=Ae^{2t}\cos t+Be^{2t}\sin t$

$Y'(t)=2Ae^{2t}\cos t-Ae^{2t}\sin t+2Be^{2t}\sin t+Be^{2t}\cos t$

$Y''(t)=4Ae^{2t}\cos t-2Ae^{2t}\sin t-2Ae^{2t}\sin t-Ae^{2t}\cos t+4Be^{2t}\sin t+2Be^{2t}\cos t+2Be^{2t}\cos t$
$-Be^{2t}\sin t$

3 cont'd

②

$4Ae^{2t}\cos t - 2Ae^{2t}\sin t - 2Ae^{2t}\sin t - Ae^{2t}\cos t + 4Be^{2t}\sin t$
$+ 2Be^{2t}\cos t + 2Be^{2t}\cos t - Be^{2t}\sin t - 2(2Ae^{2t}\cos t - Ae^{2t}\sin t$
$+ 2Be^{2t}\sin t + Be^{2t}\cos t) - 3(Ae^{2t}\cos t + Be^{2t}\sin t) = 3e^{2t}\cos t$

$(\cancel{4A} - \cancel{A} + 2B + \cancel{2B} - 4A - \cancel{2B} - \cancel{3A})\cos t + (-2A - \cancel{2A} + \cancel{4B} - B + \cancel{2A} - \cancel{4B} - 3B)\sin t$

$(-4A + 2B)\cos t + (-2A - 4B)\sin t = 3\cos t$

$-4A + 2B = 3$
$4A + 8B = 0$
$10B = 3$
$B = 3/10$

$\dfrac{(-2A - 4B = 0)}{2}$ -2
$-A = 2B$
$A = -2(3/10) = -\frac{3}{5}$

$y_p(t) = -\frac{3}{5}e^{2t}\cos t + \frac{3}{10}e^{2t}\sin t$

$y(t) = c_1e^{3t} + c_2e^{-t} - \frac{3}{5}e^{2t}\cos t + \frac{3}{10}e^{2t}\sin t$

4. $y'' - 2y' - 3y = 3te^{-t}$ $\qquad y_c(t) = c_1e^{3t} + c_2e^{-t}$ (see above)

$Y(t) = (At^2 + Bt)e^{-t}$ $\quad Y'(t) = (2At + B)e^{-t} - (At^2 + Bt)e^{-t}$

$Y''(t) = (2A)e^{-t} - (2At + B)e^{-t} - (2At + B)e^{-t} + (At^2 + Bt)e^{-t}$

$-4At - 2B + 2A + At^2 + Bt - 4At - 2B + 2At^2 + 2Bt - 3At^2 - 3Bt = 3t$

$-2B + 2A - 2B = 0$ $\qquad 2A - 4B = 0 \to 2A = 4B \Rightarrow A = 2B$

$-4A + B - 4A - 2B + 2B - 3B = 3 \Rightarrow -8A - 2B = 3$
$A + 2A - 3A = 0 \Rightarrow 0 = 0$

$-8(2B) - 2B = 3 \Rightarrow -16B - 2B = 3$
$\Rightarrow -18B = 3 \Rightarrow B = -1/6$
$A = -1/3$

$y_p(t) = -\frac{1}{3}t^2e^{-t} - \frac{1}{6}te^{-t}$

$y(t) = c_1e^{3t} + c_2e^{-t} - \frac{1}{3}t^2e^{-t} - \frac{1}{6}te^{-t}$

5. $y'' + 9y = t^2 + 3\sin t$ $\qquad y_1 = \cos(3t) \quad y_2 = \sin(3t)$

$r^2 + 9 = 0$ $\qquad y_c(t) = c_1\cos 3t + c_2\sin 3t$

$r = \pm 3$

5 cont'd

$Y(t) = At^2 + Bt + C + D\cos t + E\sin t$

$Y'(t) = 2At + B - D\sin t + E\cos t$

$Y''(t) = 2A - D\cos t - E\sin t$

$2A - D\cos t - E\sin t + 9At^2 + 9Bt + 9C + 9D\cos t + 9E\sin t = t^2 + 3\sin t$

$2A + 9C = 0$ constants $\Rightarrow \frac{2}{9} + 9C = 0 \Rightarrow 9C = -\frac{2}{9} \Rightarrow C = -\frac{2}{81}$

$9B = 0$ t $\Rightarrow B = 0$

$9A = 1$ t^2 $\Rightarrow A = 1/9$

$-D + 9D = 0$ $\cos t$ $\Rightarrow 8D = 0 \Rightarrow D = 0$

$-E + 9E = 3$ $\sin t$ $\Rightarrow 8E = 3 \Rightarrow E = \frac{3}{8}$

$Y_p(t) = 1/9\, t^2 - \frac{2}{81} + \frac{3}{8}\sin t$

$y(t) = c_1 \cos 3t + c_2 \sin 3t + 1/9\, t^2 - \frac{2}{81} + \frac{3}{8}\sin t$

6. $y'' + 9y = 5\sin(3t)$ $\qquad y_c(t) = c_1\cos(3t) + c_2\sin(3t)$ (see above)

$Y(t) = At\cos 3t + Bt\sin 3t$

$Y'(t) = A\cos 3t - 3At\sin 3t + B\sin 3t + 3Bt\cos 3t$

$Y''(t) = -3A\sin 3t - 3A\sin 3t - 9At\cos 3t + 3B\cos 3t + 3B\cos 3t - 9Bt\sin 3t$

$-6A\sin 3t - \cancel{9At\cos 3t} + 6B\cos 3t - \cancel{9Bt\sin 3t} + \cancel{9t\cos 3t} + \cancel{9Bt\sin 3t} =$

$-6A\sin 3t + 6B\cos 3t = 5\sin 3t$

$-6A = 5$ $\qquad 6B = 0 \Rightarrow B = 0$

$A = -5/6$

$Y_p(t) = -5/6\, t\cos 3t$

$y(t) = c_1\cos 3t + c_2\sin 3t - 5/6\, t\cos 3t$

7. $y'' + 2y = 4\sin 2t$ $\qquad y_1 = \cos\sqrt{2}t \quad y_2 = \sin\sqrt{2}t$

$r^2 + 2 = 0$ $\qquad y_c(t) = c_1\cos\sqrt{2}t + c_2\sin\sqrt{2}t$

$r = \pm\sqrt{2}i$

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9. $y'' + y' + 4y = 2\sinh t$

$y'' + y' + 4y = 0$

$r^2 + r + 4 = 0$

$r = \frac{-1 \pm \sqrt{1 - 4(4)}}{2} = \frac{-1 \pm \sqrt{15}\,i}{2}$

$y_1 = e^{-t/2}\cos\left(\frac{\sqrt{15}}{2}t\right)$

$y_2 = e^{-t/2}\sin\left(\frac{\sqrt{15}}{2}t\right)$

$y_c(t) = c_1 e^{-t/2}\cos\left(\frac{\sqrt{15}}{2}t\right) + c_2 e^{-t/2}\sin\left(\frac{\sqrt{15}}{2}t\right)$

$Y(t) = A\cosh t + B\sinh t \qquad Y'(t) = A\sinh t + B\cosh t$

$Y''(t) = A\cosh t + B\sinh t$

$A\cosh t + B\sinh t + A\sinh t + B\cosh t + 4A\cosh t + 4B\sinh t = 2\sinh t$

$(A + B + 4A)\cosh t = 0 \implies 5A + B = 0 \Rightarrow B = -5A$

$(B + A + 4B)\sinh t = 2 \qquad A + 5B = 2 \implies A + 5(-5A) = A - 25A = -24A = 2$

$A = -1/12 \quad B = 5/12$

$y_p(t) = -1/12\cosh t + 5/12\sinh t$

$y(t) = c_1 e^{-t/2}\cos\left(\frac{\sqrt{15}}{2}t\right) + c_2 e^{-t/2}\sin\left(\frac{\sqrt{15}}{2}t\right) - 1/12\cosh t + 5/12\sinh t$

10. $y'' - y' - 2y = \cosh(2t) = \frac{1}{2}e^{2t} + \frac{1}{2}e^{-2t}$

$y'' - y' - 2y = 0$

$r^2 - r - 2 = 0$

$(r-2)(r+1) = 0$

$r = 2,\ r = -1$

$y_1 = e^{2t},\ y_2 = e^{-t}$

$y_c(t) = c_1 e^{2t} + c_2 e^{-t}$

$Y(t) = Ate^{2t} + Be^{-2t} \qquad Y'(t) = Ae^{2t} + 2Ate^{2t} - 2Be^{-2t}$

$Y''(t) = 2Ae^{2t} + 2Ae^{2t} + 4Ate^{2t} + 4Be^{-2t} = 4Ae^{2t} + 4Ate^{2t} + 4Be^{-2t}$

$4Ae^{2t} + 4Ate^{2t} + 4Be^{-2t} - Ae^{2t} - 2Ate^{2t} + 2Be^{-2t} - 2Ate^{2t} - 2Be^{-2t} =$

$te^{2t}\overset{0}{(\cancel{4A - 2A - 2A})} + e^{2t}\underbrace{(4A - A)}_{3A} + e^{-2t}(4B + \cancel{2B} - \cancel{2B}) = \frac{1}{2}e^{2t} + \frac{1}{2}e^{-2t}$

$3A = 1/2 \Rightarrow A = 1/6 \qquad 4B = 1/2 \implies B = \frac{1}{8}$

$y_p(t) = 1/6\,te^{2t} + 1/8\,e^{-2t}$

$y(t) = c_1 e^{2t} + c_2 e^{-t} + 1/6\,te^{2t} + 1/8\,e^{-2t}$

11. $y'' + y' - 2y = 2t$ $\quad y(0)=0, y'(0)=1$

$y_c(t) = c_1 e^{2t} + c_2 e^{-t}$ (see above)

$Y(t) = At + B \qquad Y'(t) = A \qquad Y''(t) = 0$

$0 + A - 2At - 2B = 2t \Rightarrow -2A = 2 \Rightarrow A = -1, \quad A - 2B = 0$

$\Rightarrow -1 - 2B = 0 \Rightarrow -2B = 1$

$\Rightarrow B = -\frac{1}{2}$

$y_p(t) = -t - \frac{1}{2}$

$y(t) = c_1 e^{2t} + c_2 e^{-t} - t - \frac{1}{2}$

$0 = c_1 + c_2 - \frac{1}{2} \Rightarrow c_1 + c_2 = \frac{1}{2}$

$y'(t) = 2c_1 e^{2t} - c_2 e^{-t} - 1$ $\quad +$

$1 = 2c_1 - c_2 - 1 \Rightarrow 2c_1 - c_2 = 2$

$3c_1 = \frac{5}{2} \Rightarrow c_1 = \frac{5}{6}$

$\Rightarrow \frac{5}{6} + c_2 = \frac{1}{2} \Rightarrow c_2 = \frac{1}{2} - \frac{5}{6} = \frac{3}{6} - \frac{5}{6} = -\frac{1}{3}$

$y(t) = \frac{5}{6} e^{2t} - \frac{1}{3} e^{-t} - t - \frac{1}{2}$

12. $y'' + 4y = 3\sin 2t$ $\quad y(0) = 2, \; y'(0) = -1$

$r^2 + 4 = 0 \qquad y_1 = \cos 2t \qquad y_2 = \sin 2t$

$r = \pm 2i \qquad y_c(t) = c_1 \cos 2t + c_2 \sin 2t$

$Y(t) = At\cos 2t + Bt\sin 2t$

$Y'(t) = A\cos 2t - 2At\sin 2t + B\sin 2t + 2Bt\cos 2t$

$Y''(t) = -2A\sin 2t - 2A\sin 2t - 4At\cos 2t + 2B\cos 2t + 2B\cos 2t - 4Bt\sin 2t$

$= -4A\sin 2t - 4At\cos 2t + 4B\cos 2t - 4Bt\sin 2t$

$-4A\sin 2t \cancel{- 4At\cos 2t} + 4B\cos 2t \cancel{- 4Bt\sin 2t} + \cancel{4At\cos 2t} + \cancel{4Bt\sin 2t}$

$= -4A\sin 2t + 4B\cos 2t = 3\sin 2t$

$\Rightarrow B = 0 \qquad -4A = 3 \Rightarrow A = -\frac{3}{4}$

$y_p(t) = -\frac{3}{4}t\cos 2t$

$y(t) = c_1\cos 2t + c_2\sin 2t - \frac{3}{4}t\cos 2t$

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$2 = c_1 + c_2(0) - 0 \Rightarrow c_1 = 2$

$y'(t) = -2c_1 \sin 2t + 2c_2 \cos 2t - \frac{3}{4}\cos 2t + \frac{3}{2}t \sin 2t$

$-1 = 0 + 2c_2 - \frac{3}{4} + 0 \Rightarrow -\frac{1}{4} = 2c_2 \Rightarrow c_2 = -\frac{1}{8}$

$y(t) = 2\cos 2t - \frac{1}{8}\sin 2t - \frac{3}{4}t\cos 2t$

13. $y'' + y = t + t\sin t$

$r^2 + 1 = 0$

$r = \pm i$

$y_1 = \cos t \quad y_2 = \sin t$

$y_c(t) = c_1\cos t + c_2 \sin t$

$Y(t) = At + B + (Ct^2 + Dt)\cos t + (Et^2 + Ft)\sin t$ *

$Y'(t) = A + (2Ct + D)\cos t - (Ct^2 + Dt)\sin t + (Et^2 + Ft)\cos t + (2Et + F)\sin t$

$Y''(t) = (2C)\cos t - (2Ct + D)\sin t - (2Ct + D)\sin t - (Ct^2 + Dt)\cos t +$
$(2Et + F)\cos t - (Et^2 + Ft)\sin t + (2E)\sin t + (2Et + F)\cos t =$

$2C\cos t + (-4Ct - 2D)\sin t + \cancel{(-Ct^2 - Dt)\cos t} + (4Et + 2F)\cos t +$
$\cancel{(-Et^2 - Ft)\sin t} + 2E\sin t + At + B + \cancel{(Ct^2 + Dt)\cos t} + \cancel{(Et^2 + Ft)\sin t}$

$= t + t\sin t \Rightarrow$

$A = 1 \quad B = 0$

$(2C + 4Et + 2F)\cos t + (-4Ct - 2D + 2E)\sin t = t\sin t$

$E = 0 \quad 2C + 2F = 0 \Rightarrow C = -F \qquad -4C = 0 \Rightarrow C = 0 \Rightarrow F = 0$

$-2D = 1 \Rightarrow D = -\frac{1}{2}$

$y_p(t) = t - \frac{1}{2}t\cos t$

$y(t) = c_1\cos t + c_2\sin t + t - \frac{1}{2}t\cos t$

* if you can't make $Ct\cos t + Dt\sin t$ work out as you need it to, guessing additional terms to check solution works okay, since the unneeded terms will go to zero in the end.