Singular Points

A **singular point** is a term for a point value of a variable which makes a function undefined at that value (and not on any part of the open interval around that value). A singular point, for instance, is typically a point for which the denominator of the function is equal to zero. If the denominator function is continuous and smooth, this will happen at only a single point, or at a set of discrete single points. In the case of polynomials, this set will be finite, however, it could be infinite, if the denominator function in question is a trigonometric function, for instance. All other points for which the function is defined (where the denominator is not equal to zero), are called **ordinary points**.

Let us look at some examples.

Example 1. In the function $p(x) = \frac{x-2}{x^3-x}$, locate any singular points.

Here, we have a simple rational expression, so to find the singular points, we must determine where the denominator could be zero.

$$x^3 - x = x(x^2 - 1) = x(x-1)(x+1) = 0$$

We can see that if then allow each factor to be equal to zero, we obtain the values x=0, x= -1, x=1 as our singular points. All other values of x are ordinary points.

Example 2. Find any singular points in the function $q(x) = \frac{x}{\sin(x)}$.

Again, since we are dealing with some kind of fraction, we'll set the denominator equal to zero.

$$\sin(x) = 0$$

But for what values of x is this true? Any multiple of π will work. This is an infinite set of values, but each point is discrete. Our original q function is perfectly well defined everywhere except at these discrete points. All these other points are ordinary points, but $x=k\pi$, (k=0, ±1, ±2,…) are all singular points.

Example 3. Find the singular points of the function $r(x) = \frac{\tan(x)}{\ln(x)}$.

Let us consider the denominator first. $\ln(x)$ is defined for values greater than zero. For any value on the interval $(-\infty, 0]$, the denominator and thus the entire expression are not defined. These are not singular points, however, since this is a continuous interval and not a discrete point. There is one other place that the denominator is zero, however, that is when x=1, since *ln(1)=0*. This is a discrete point, and so, x=1 is a singular point.

We aren't done yet, though. In the previous examples, the functions in the numerator were defined everywhere and so we did not need to consider them. Not so in this example. $\tan(x)$

itself is undefined for some values of x and so we must look at these as well. Since $\tan(x) = \frac{\sin(x)}{\cos(x)}$ we must consider the values of x where $\cos(x) = 0$. This occurs for odd multiples of $\frac{\pi}{2}$ or $x = \frac{(2k+1)\pi}{2}$ $(k = 0 \pm 1, \pm 2, \ldots)$. These values are also singular points since they are discrete values of x.

Practice Problems. Find all the singular points in each of these functions below. For each differential equation, a singular point of any coefficient function (when the equation is in standard form) is a singular point of the entire equation, even if the point is only undefined in one of the functions.

1. $p(x) = \frac{x+1}{x^2}$
2. $q(x) = 6\cot(x)$
3. $r(x) = \frac{1}{1-e^{-x/2}}$
4. $s(x) = \frac{x}{\sin x}$
5. $y'' + \frac{2}{x}y' + \frac{x+1}{2x^2}y = 0$
6. $y'' + \frac{2x}{1+2x}y' - \frac{3}{1+2x}y = 0$
7. $x^2(x-3)^2y'' + (x-3)y' + 5x^2y = 0$
8. $(x\sin x)y'' + (\cos x)\, y' + (\ln x)y = 0$

In differential equations, particularly when working with series solutions, we prefer to solve around ordinary points when we can because we avoid complications associated with singular points. This is not always possible and there are circumstances where we may wish to work at the point that is problematic (singular). To know whether or not we can, we need one further distinction: that between a **regular** singular point, and an **irregular** one. (This is a topic that also comes up in other topics such as working with complex variables.)

Defining a regular singular point is a bit tricky, but an irregular singular is easy: singular points are irregular if they are not regular. But let's look at the definition of regular singular points. I am going to define them here in the context of a differential equation since that will be our application for them, but the definition is similar adjusted for other contexts.

Consider the second order linear differential equation in standard form: $y'' + p(x)y' + q(x)y = g(x)$. The point c is a regular singular point if all three of the following conditions hold:

1) c is a singular point for $p(x)$ or $q(x)$.
2) $(x-c)p(x)$ the limit exists at c (i.e. the reduced form could be defined, or a technique like L'Hôpital's should be able to resolve the limit to a finite value).
3) $(x-c)^2q(x)$ the limit exists at c (i.e. the reduced form is defined at c, or apply L'Hôpital's Rule and obtain a finite value; put another way, the point must be a hole and not an asymptote).

Each singular point must be classified separately, so it is possible for some of them to be regular and some to be irregular in the same differential equation.

It should also be noted here that as the number of derivatives in the function goes up, so do the conditions. For instance in the third order linear differential equation $y''' + p(x)y'' + q(x)y' + r(x)y = g(x)$ would add the fourth condition that $(x-c)^3 r(x)$ would need to be defined at c. The further away you get from the leading derivative, the more powers of (*x*-*c*) you are allowed to try to clear the singularity in the denominator.

In particular, you will note that regular singular points are defined by asymptotes of rational expressions and zeros of the polynomials in the denominators. If the singular points come from non-rational expressions (like trigonometric functions, for example) this procedure of multiplying by powers of (*x*-*c*) will sometimes have no effect as with $\frac{x}{1-cos\,x}$, or they may produce special cases like $\frac{x}{\sin x}$. If the expression has a limit which is not finite, these points will be irregular.

Example 4. Consider the differential equation $y'' + \frac{x^2+3}{x(x-1)^2}y' + \frac{x+1}{x^2(x-1)}y = 0$.

Here we identify $p(x) = \frac{x^2+3}{x(x-1)^2}$ and $q(x) = \frac{x+1}{x^2(x-1)}$. Both functions are undefined at *x*=0, and *x*=1, and so both these points are singular points. Let us set about classifying them.

i) For the case of *c*=0, the factor we will use to test for regularity is (*x*-0) or just *x*.

$$xp(x) = x\frac{x^2+3}{x(x-1)^2} = \frac{x^2+3}{(x-1)^2}$$
$$x^2 q(x) = x^2\frac{x+1}{x^2(x-1)} = \frac{x+1}{(x-1)}$$

Now consider the point *x*=0. Are these functions defined at this point now? Yes, because the factors of *x* were successfully cancelled in the denominators, so the reduced form is defined at *x*=0, so we don't have to go on to advanced techniques to check the limit here. So *x*=0 satisfies the conditions of a regular singular point.

ii) Let's now consider the case of *c*=1, using the factor (*x*-1):

$$(x-1)p(x) = (x-1)\frac{x^2+3}{x(x-1)^2} = \frac{x^2+3}{x(x-1)}$$
$$(x-1)^2 q(x) = (x-1)^2\frac{x+1}{x^2(x-1)} = \frac{(x+1)(x-1)}{x^2}$$

This procedure produced an extra factor of (*x*-1) in the numerator of the second test, but this doesn't matter because it did successfully eliminate the factor in the denominator of this expression. What about the first one though? There is still a factor of (*x*-1) left in the denominator, and so the expression is still undefined. Since there is a zero in the denominator, but not in the numerator, or advanced limit techniques will not apply. This means that *x*=1 is an irregular singular point since it failed our test for regularity.

Thus, this equation has one regular singular point at *x*=0, one irregular singular point at *x*=1, and all other points are ordinary.

Example 5. Determine if there are any singular points in the differential equation $y'' + \frac{1}{x\ln x}y' + \frac{x}{\sin x}y = 0$. Classify each singular point as regular or irregular.

Here, our $p(x) = \frac{1}{x\ln x}$ and $q(x) = \frac{x}{\sin x}$. $p(x)$ is undefined at *x*=0, and *x*=1. And $q(x)$ is undefined at all multiples of π including 0. Thus, our set of singular points is $x = \{0, 1, \pm\pi, \pm 2\pi, , \dots\}$. The multiples of π will all behave the same, but we will consider them last. Let's start with *x*=0.

i) As in Example 4, we will multiply by *x* to test our regularity conditions.

$$xp(x) = x\frac{1}{x\ln x} = \frac{1}{\ln x}$$
$$x^2 q(x) = x^2\frac{x}{\sin x} = \frac{x^3}{\sin x}$$

Our test with *p*(*x*) worked in the sense that it cancelled the *x* factor, but *ln(x)* is still undefined at *x*=0. In any case, let's consider *q*(*x*) anyway: the expression is still undefined at *x*=0. So what we need to do is check the limit to see if it exists and is finite:

$$\lim_{x\to 0}\frac{x^3}{\sin x} = \left(\lim_{x\to 0}\frac{x}{\sin x}\right)\left(\lim_{x\to 0}x^2\right) = (1)(0) = 0$$

Our limit does exist at *x*=0, so *x*=0 is an irregular singular point (because the test with *p(x)* failed).

ii) We'll use the factor (*x*-1) to test the point *c*=1.

$$(x-1)p(x) = (x-1)\frac{1}{x\ln x} = \frac{x-1}{x\ln x}$$
$$(x-1)^2 q(x) = (x-1)^2\frac{x}{\sin x} = \frac{x(x-1)^2}{\sin x}$$

Our *q* expression was never a problem at *x*=1, so this is fine, but our *p* expression is still undefined at *x*=1 since the factor (*x*-1) cannot cancel with all or part of *ln*(*x*). Therefore, we need to check the limit:

$$\lim_{x\to 1}\frac{x-1}{x\ln x}=\lim_{x\to 1}\frac{1}{\ln x+1}=\frac{1}{0+1}=1$$

Using L'Hôpital's rule, we find that the limit does exist and is finite, so x=1 is a regular singular point.

iii) Let's finally look at these multiples of π. We'll consider the factor (x-π) as the simplest case since we can expect all these factors to remain the same.

$$(x-\pi)p(x)=(x-\pi)\frac{1}{x\ln x}=\frac{x-\pi}{x\ln x}$$
$$(x-\pi)^2q(x)=(x-\pi)^2\frac{x}{\sin x}=\frac{x(x-\pi)^2}{\sin x}$$

This is a bit of the reverse of our previous results at x=1, but the conclusion is the same. The p expression never had a problem at π, so all is well. But our q expression remains undefined at π since (x-π) and *sin*(x) cannot cancel with each other, but the expression is a form of 0/0, so we need to apply L'Hôpital's rule to see what is happening.

$$\lim_{x\to\pi}\frac{x(x-\pi)^2}{\sin x}=\lim_{x\to\pi}\frac{(x-\pi)^2+2x(x-\pi)}{\cos x}=\frac{0+2\pi(0)}{-1}=0$$

The result will be repeated for all possible multiples of π, and so we conclude that they are all regular singular points.

Example 6. Find the singular points (if any) of $y''+\left(\frac{1}{1-cosx}\right)y'+(\cot x)y=0$ and determine if they are regular or irregular.

Our singular points are going to occur at all multiples of π since that's where *cot(x)* will be undefined there. The expression $p(x)=\frac{1}{1-\cos x}$ will be undefined when *cos(x)*=1, or only even multiples of π.

i) Let's consider the case of x=0.

$$xp(x)=\frac{x}{1-\cos x}$$
$$x^2q(x)=x^2\cot(x)$$

In both of these cases, we'll have to check the limit using L'Hôpital's rule or employing some special limits.

$$\lim_{x\to 0}\frac{x}{1-\cos x}=\lim_{x\to 0}\frac{1}{\sin x}=\frac{1}{0}$$
$$\lim_{x\to 0}x^2\cot(x)=\lim_{x\to 0}\frac{x^2\cos(x)}{\sin x}=\frac{(0)(1)-(0)(0)}{(1)}=0$$

Our *q(x)* term reduced to a defined limit nicely, but the *p(x)* term did not. The behaviour here is still asymptotic, so the point x=0 is an irregular singular point.

ii) Use the model of Examples 4 and 5 to examine the points $x=\pi$, and $x=2\pi$. What can you generalize about the rest of the points based on these examples?

Practice Problems. Find all the singular points of the differential equations below and determine whether each one is regular or irregular.

1. $y'' + \frac{(9-x^2)}{x(1-x)^2} y' + \frac{2+5x}{x^2(1-x)^2} y = 0$
2. $y'' + \frac{1}{x} y' + \frac{1}{x^3} y = 0$
3. $x^3 y'' - (1-\cos x)y' + xy = 0$
4. $x^2(x-3)^2 y'' + (x-3)y' + 5x^2 y = 0$
5. $x^2 y'' + (6x + x^2)y' + xy = 0$
6. $x^2(x-2)^2 y'' + \sin(x)\, y' + (2-x)y = 0$
7. $3x(x-2)^2 y'' + 2xy' + (x-2)y = 0$
8. Finish part (ii) in Example 6.
9. $(1-\ln x)y'' + 2xy' + (xln^2 x)y = 0$
10. $xy'' + 4(\csc x)y' + (\cot x)y = 0$
11. $(\sin^2 x)\, y'' - 5e^x y' + \frac{x}{e^x - 1} y = 0$
12. $(x^2+1)x^2 y'' + 2(x^4-1)y' - 12(x-1)^3 y = 0$