Differential Equations Bernoulli Equations Key

1. $y' + 3x^2 y = x^2 y^3$ $n = 3$ $(1-n)y^{-n} = -2y^{-3}$

$-2y^{-3}y' - 6x^2 y^{-2} = -2x^2$ $z = y^{-2}$

$z' - 6x^2 z = -2x^2$ $z' = -2y^{-3}y'$

$e^{-2x^3}z' - 6x^2 e^{-2x^3} z = -2x^2 e^{-2x^3}$ $\mu = e^{\int -6x^2\,dx} = e^{-2x^3}$

$\int (e^{-2x^3} z)' = \int -2x^2 e^{-2x^3}$ $u = -2x^3$

$e^{2x^3} \cdot e^{-2x^3} z = \left(\frac{1}{3}e^{-2x^3} + C\right)e^{2x^3}$ $\frac{1}{3}du = \frac{-6x^2}{3} = -2x^2$

$z = \frac{1}{3} + Ce^{2x^3}$

$y^{-2} = \frac{1}{3} + Ce^{2x^3} \Rightarrow y^2 = \frac{1}{\frac{1}{3} + Ce^{2x^3}} \Rightarrow \boxed{y = \pm\sqrt{\frac{3}{1 + Ce^{2x^3}}}}$

2. $y' + xy = xy^{-1}$ $n = -1$ $(1-n)y^{-n} = 2y$

$2yy' + 2xy^2 = 2x$ $z = y^2$ $z' = 2yy'$

$z' + 2xz = 2x$ $\mu = e^{\int 2x\,dx} = e^{x^2}$

$e^{x^2}z' + 2xe^{x^2}z = 2xe^{x^2}$

$\int (e^{x^2}z)' = \int 2xe^{x^2}dx = e^{x^2} + C$

$e^{-x^2}e^{x^2}z = (e^{x^2} + C)e^{-x^2} \Rightarrow z = 1 + Ce^{-x^2} \Rightarrow y^2 = 1 + Ce^{-x^2}$

$\boxed{y = \pm\sqrt{1 + Ce^{-x^2}}}$

3. $y' + \frac{y}{x} = xy^2$ $n = 2$ $(1-n)y^{-n} = -1y^{-2}$

$-y^{-2}y' - \frac{1}{x}y^{-1} = -x$ $z = y^{-1}$ $z' = -y^{-2}y'$

$z' - \frac{1}{x}z = -x$ $\mu = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln 1/x} = \frac{1}{x}$

$\frac{1}{x}z' - \frac{1}{x^2}z = -1$

3 cont'd.

$$\int \left(\tfrac{1}{x} z\right)' = \int -1\,dx = -x + C$$

$$x \cdot \tfrac{1}{x} z = (-x+C)x \implies z = -x^2 + Cx \implies y^{-1} = -x^2 + Cx$$

$$\boxed{y = \frac{1}{Cx - x^2}}$$

4. $y' + \frac{y}{x} = x\sqrt{y}$ $\qquad n = 1/2 \qquad (1-n)y^{-n} = \tfrac{1}{2}y^{-1/2}$

$$\tfrac{1}{2}y^{-1/2}y' + \tfrac{1}{2x}y^{1/2} = \tfrac{1}{2}x \qquad z = y^{1/2} \quad z' = \tfrac{1}{2}y^{-1/2}y'$$

$$z' + \tfrac{1}{2x}z = \tfrac{1}{2}x \qquad \mu = e^{\int \frac{1}{2x}dx} = e^{\frac{1}{2}\ln x} = e^{\ln\sqrt{x}} = \sqrt{x}$$

$$x^{1/2}z' + \tfrac{1}{2}x^{-1/2}z = \tfrac{1}{2}x^{1/2}$$

$$\int (x^{1/2}z)' = \int \tfrac{1}{2}x^{1/2}dx = \tfrac{1}{2}\cdot\tfrac{2}{3}x^{3/2} + C$$

$$x^{-1/2}x^{1/2}z = \left(\tfrac{1}{3}x^{3/2} + C\right)x^{-1/2} \implies z = \tfrac{1}{3}x + Cx^{-1/2}$$

$$y^{1/2} = \tfrac{1}{3}x + Cx^{-1/2} \implies \boxed{y = \left(\tfrac{1}{3}x + Cx^{-1/2}\right)^2}$$

5. $\frac{yy'}{y} - \frac{2y^2}{y} = \frac{e^x}{y} \implies y' - 2y = e^x y^{-1} \qquad n = -1$

$$(1-n)y^{-n} = 2y'$$

$$2yy' - 4y^2 = 2e^x \qquad z = y^2 \quad z' = 2yy'$$

$$z' - 4z = 2e^x \qquad \mu = e^{\int -4\,dx} = e^{-4x}$$

$$e^{-4x}z' - 4e^{-4x}z = 2e^x \cdot e^{-4x} = 2e^{-3x}$$

$$\int (e^{-4x}z)' = \int 2e^{-3x}dx = -\tfrac{2}{3}e^{-3x} + C$$

$$z = \left(-\tfrac{2}{3}e^{-3x} + C\right)e^{4x} = -\tfrac{2}{3}e^x + Ce^{4x}$$

$$y^2 = -\tfrac{2}{3}e^x + Ce^{4x} \implies \boxed{y = \pm\sqrt{-\tfrac{2}{3}e^x + Ce^{4x}}}$$

I used integrating factors for the first five problems.
I will use variation of parameters for the remaining five.

6. $y' + xy = xe^{-x^2} y^{-3}$ $\qquad n = -3 \quad (1-n)y^{-n} = 4y^3$

$4y^3 y' + 4x y^4 = 4xe^{-x^2}$ $\qquad z = y^4 \quad z' = 4y^3 y'$

$z' + 4xz = 4xe^{-x^2}$ $\qquad \mu = e^{\int 4x\,dx} = e^{2x^2}$

$$z = e^{-2x^2} \int 4xe^{-x^2} \cdot e^{2x^2}\,dx = e^{-2x^2} \int 4xe^{x^2}\,dx =$$

$$e^{-2x^2}\left[2e^{x^2} + C\right] = 2e^{-x^2} + Ce^{-2x^2}$$

$$y^4 = 2e^{-x^2} + Ce^{-2x^2} \Rightarrow y = \sqrt[4]{2e^{-x^2} + Ce^{-2x^2}}$$

7. $y' + y = xy^2$ $\qquad n = 2 \quad (1-n)y^{-n} = -1y^{-2}$

$-y^{-2}y' - y^{-1} = -x$ $\qquad z = y^{-1} \quad z' = -1y^{-2}y'$

$z' - z = -x$ $\qquad \mu = e^{\int -1\,dx} = e^{-x}$

$$z = e^x \int e^{-x} \cdot x\,dx = -e^x \int xe^{-x}\,dx$$

±	u	dv
+	x	e^{-x}
−	1	$-e^{-x}$
+	0	e^{-x}

$$= -e^x\left[-xe^{-x} - e^{-x} + C\right] =$$

$$x + 1 + Ce^x$$

$$y^{-1} = x + 1 + Ce^x \Rightarrow \boxed{y = \frac{1}{x+1+Ce^x}}$$

8. $\frac{dy}{dx} + 2xy = xy^2$ $\qquad n = 2 \quad (1-n)y^{-n} = -1y^{-2}$

$-1y^{-2}y' - 2xy^{-1} = -x$ $\qquad z = y^{-1} \quad z' = -y^{-2}y'$

$z' - 2xz = -x$ $\qquad \mu = e^{\int -2x\,dx} = e^{-x^2}$

$$z = e^{x^2} \int -xe^{-x^2}\,dx = e^{x^2}\left[\tfrac{1}{2}e^{-x^2} + C\right] = \tfrac{1}{2} + Ce^{x^2}$$

$$y^{-1} = \tfrac{1}{2} + Ce^{x^2} \Rightarrow \boxed{y = \frac{2}{1+Ce^{x^2}}}$$

9. $\frac{dy}{dx} + \frac{y}{x} = \frac{y^3}{x^2}$ $\quad n=3 \quad (1-n)y^{-n} = -2y^{-3}$

$-2y^{-3}y' - \frac{2}{x}y^{-2} = -\frac{2}{x^2}$ $\quad z = y^{-2} \quad z' = -2y^{-3}y'$

$z' - \frac{2}{x}z = -\frac{2}{x^2}$ $\quad \mu = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = e^{\ln\frac{1}{x^2}} = \frac{1}{x^2}$

$$z = x^2\int \frac{1}{x^2}\left(-\frac{2}{x^2}\right)dx = x^2\int -2x^{-4}dx = x^2\left[\frac{-2}{-3}x^{-3} + C\right]$$

$$= x^2\left[\frac{2}{3x^3} + C\right] = \frac{2}{3x} + Cx^2$$

$$y^{-2} = \frac{2}{3x} + Cx^2 \implies \boxed{y = \pm\sqrt{\frac{3x}{2+Cx^3}}}$$

10. $\frac{x\frac{dy}{dx}}{x} + \frac{y}{x} = \frac{xy^5}{x} \implies y' + \frac{y}{x} = y^5$ $\quad n=5 \quad (1-n)y^{-n} = -4y^{-5}$

$-4y^{-5}y' - \frac{4}{x}y^{-4} = -4$ $\quad z = y^{-4} \quad z' = -4y^{-5}y'$

$z' - \frac{4}{x}z = -4$ $\quad \mu = e^{\int -\frac{4}{x}dx} = e^{-4\ln x} = e^{\ln\frac{1}{x^4}} = \frac{1}{x^4}$

$$z = x^4\int -4x^{-4}\,dx = x^4\left[\frac{-4}{-3}x^{-3} + C\right] = \frac{4}{3}x + Cx^4$$

$$y^{-4} = \frac{4}{3}x + Cx^4 \implies \boxed{y = \pm\sqrt[4]{\frac{3}{4x+Cx^4}}}$$