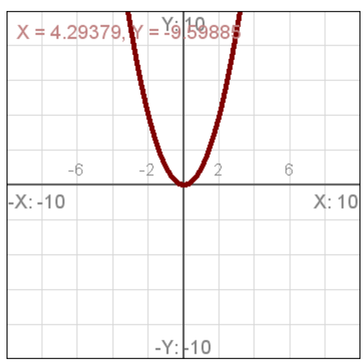
Transformations

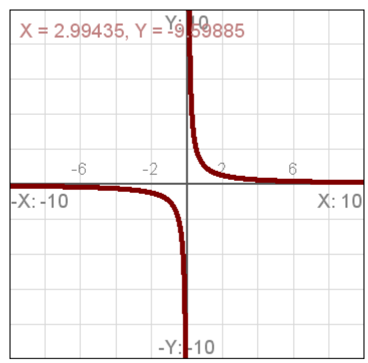
Transformations begins with the idea that all graphs of a basic form have the same general shape, and we can **transform** a standard graph into a more specific graph with certain properties by altering the graph in predictable ways. There are six ways that we can transform a graph:

i. Horizontal Stretch/Compression
ii. Horizontal Reflection
iii. Horizontal Shift
iv. Vertical Stretch/Compression
v. Vertical Reflection
vi. Vertical Shift

In general, we will wish to perform the actions in this order, performing all horizontal changes first, then all vertical ones, to avoid interaction of the parts in unfortunate ways. [However, since stretches/compressions and reflections, can be done in either order as both will involve multiplication.] In practice, with most of the functions we will be working with, we can achieve a horizontal stretch/compression by doing the opposite vertical stretch/compression, and this will sometimes be true of reflections as well. This is not always the case, as we will see when dealing with Trigonometric functions, so we will treat each one separately, even if we do de-emphasize these cases for now.

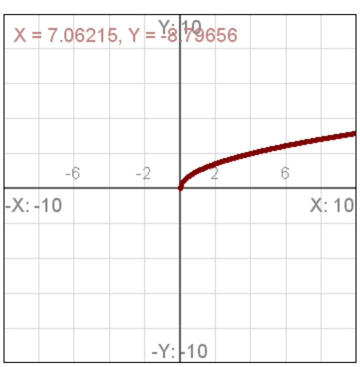
To begin any discussion of transformation, we need a basic inventory of functions (or equations). We will consider the following: parabola, reciprocal function, square root function, the circle (not a function), and the sine function. These graphs are shown in order below. Our process will work for other graphs as well, and you will be allowed to experiment on these in the problem section.

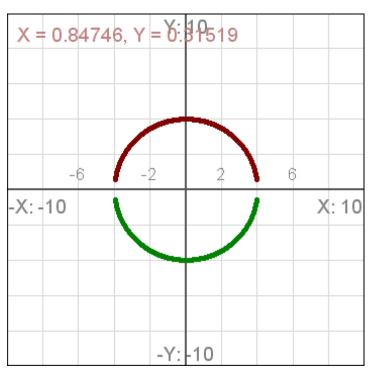


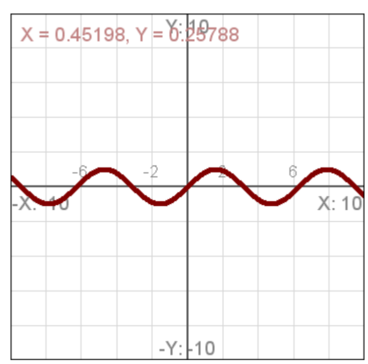


While it is a bit "out of order" it is traditional to start with the **horizontal and vertical shifts**.

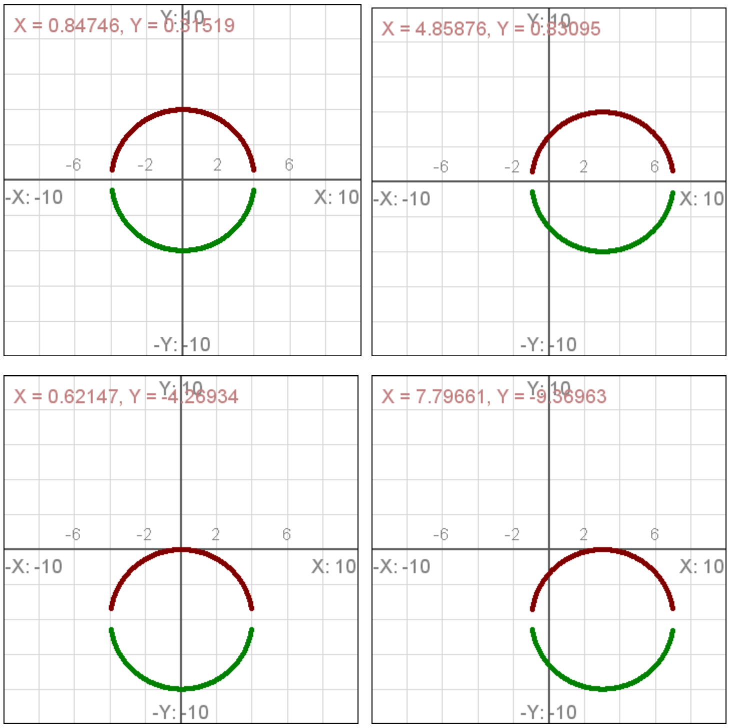
If we are not obliged to maintain function notation, there is actually a great deal that is similar about how vertical and horizontal shifts are obtained, and we can see this most easily in a non-function, the circle.





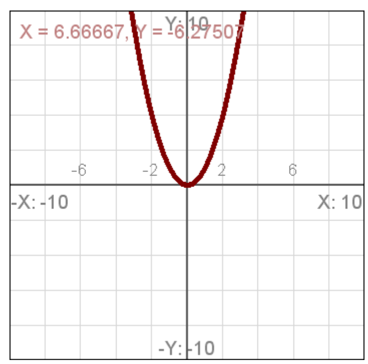


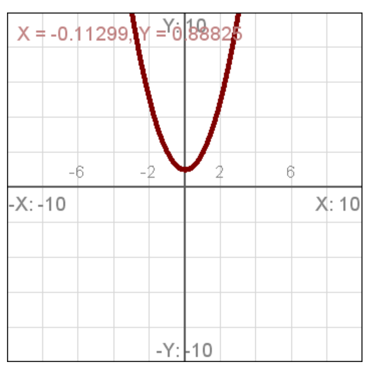
The equation of the circle, centered at the origin, is $x^2 + y^2 = r^2$. The graph above has r=4 for illustration. In the equation of the circle, if we want to shift it so that the center is now at (3,0), i.e. a

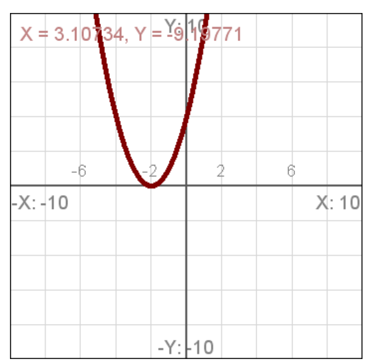


shift of positive three in the *x*-direction, our equation becomes: $(x-3)^2 + y^2 = r^2$. We subtract the amount (and direction) of the shift from the *x*-variable. This creates a horizontal shift. Compare the two graphs. Likewise, if I wanted to shift the graph into the *y*-direction, I would subtract the desired amount from the *y* variable. So, supposed I wanted to shift the circle from the orgin down to have the center at (0,-4), a vertical shift of 4 down, I would subtract -4 from *y* in the equation or $y-(-4) = y+4$ giving us $x^2 + (y+4)^2 = r^2$, as shown to the left. And of course, I can apply both shifts simultaneously, if I want the center not at (0,0), but at (3,-4). My equation then is $(x-3)^2 + (y+4)^2 = r^2$, and the graph is shown to the left. We saw this in the section on circles.

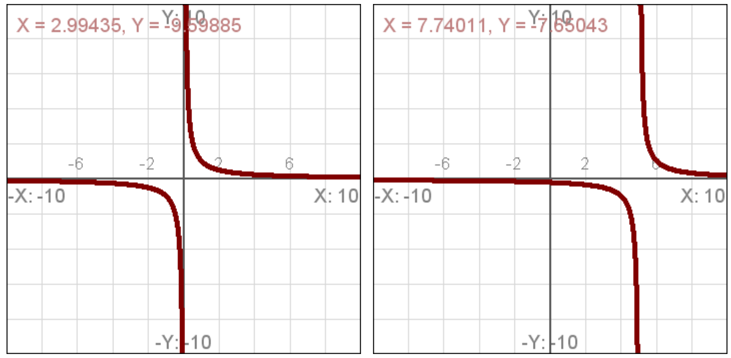
We can think of the process similarly for functions. Consider the equation $y = x^2$, our basic parabola. [Recall, that because this is a function, we will eventually want to isolate *y* so that we can write it as $f(x) = x^2$.] If we want to shift the graph horizontally 2 units left (i.e. by -2), we subtract this from the *x*-variable, and we do it inside any other operations in the equation. This gives us: $y = (x+2)^2$ since $x-(-2) = x+2$. Compare with the original graph. If we wish to shift the graph vertically, we will subtract from the *y*-variable. Let's say we want to shift it up by 1. Our equation would become $y-1 = x^2$. Which is a fine equation, but not good function notation, so if we solve for *y*, we get $y = x^2 + 1$ or $g(x) = x^2 + 1$.



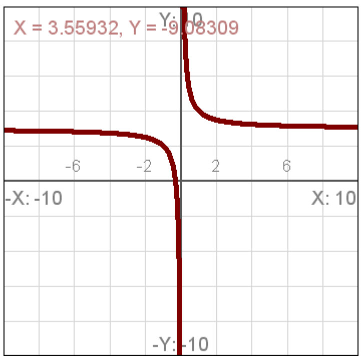




Notice that once we solve for *y*, we get the form that our textbook introduces vertical shifts using. Horizontal shifts you subtract, vertical shifts you add… but this overlooks the similarity in the two processes. Unfortunately, we do have to use these ideas when we talk about shifts using function notation.



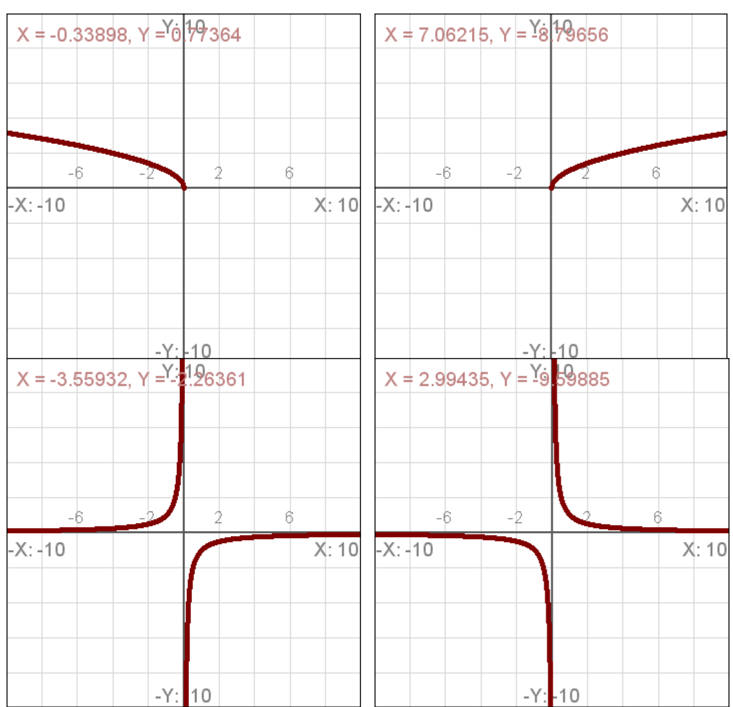
In function notation, everything that applies horizontally is done "inside" the function notation, and everything that is vertical is applied "outside" the function notation. So to represent the horizontal shift of a function $h(x)$ to the right by 5, we write $h(x-5)$, which tells us to replace *x* everywhere in our equation with *x*-5. Thus, $h(x-5)=\frac{1}{x-5}$. If we wanted to do a vertical shift of positive 3, we would write $h(x)+3$, or $h(x)+3=\frac{1}{x}+3$. Originally, without function notation, recall that this would have been $y-3=\frac{1}{x}$, and we then added three to both sides to get the new function $y=\frac{1}{x}+3$.



To apply shifts to a series of points, apply the horizontal shift to the *x*-coordinates, and the vertical shift to the *y*-coordinates of each point. When applying transformations directly to points, apply each transformation directly, i.e. if you want to move right, add, even in the *x*-direction.

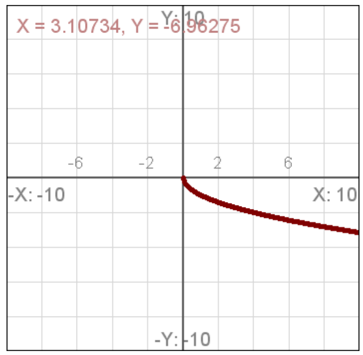
We will see this relationship again between horizontal and vertical transformations: when employing function notation, horizontal shifts will appear in the equation doing the "opposite", and vertical transformations will do straightforwardly what we expect them to do. As we've seen though, this is a direct result of all transformations really doing the "opposite" all the time in non-functions, and then our solving again for *y*, which reverses the effect.

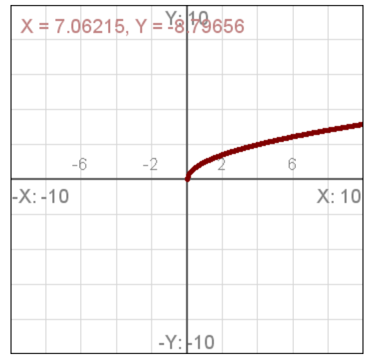
To see **horizontal reflections**, we will have to work with equations that are not symmetric. The best one for this is $y=\sqrt{x}$ (on the right) since it has no symmetry whatsoever. To apply a horizontal reflection, we are going to apply a negative to the *x*-variable in the equation (inside any operations). The horizontal reflection will reflect the graph across the *y*-axis. Thus, the square root graph, reflected across the y-axis is $y=\sqrt{-x}$ (on the left). This graph is perfectly well defined when the x-values

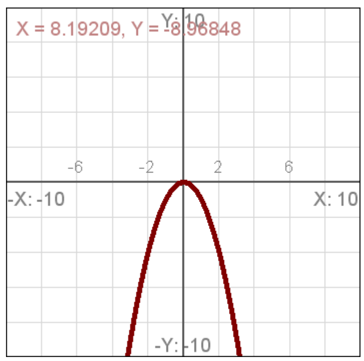


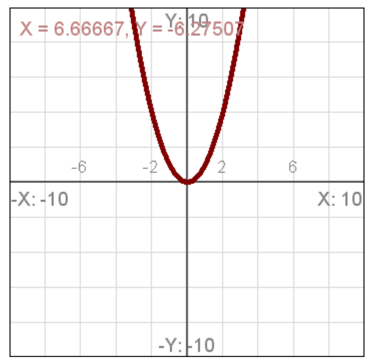
being put into the equation are already negative: the additional negative will make them all positive. We can see something similar happening in the graph of $y=\frac{1}{x}$ (on the right) compared to the graph of $y=\frac{1}{(-x)}$ (on the left). It should be noted here that graphs that are y-symmetric cannot be reflected horizontally because the graph before and the graph after looks the same.

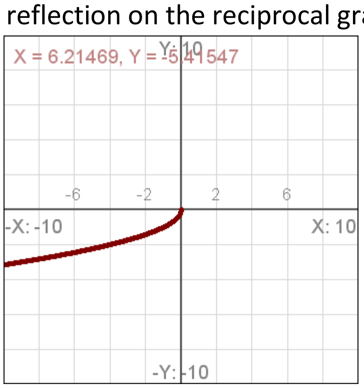
Vertical reflections are done similarly, but with the negative on the outside of the equation. (In truth, the negative is being applied to the *y*-variable, and them "moved" to the other side to isolate *y* again.) We return to the square root graph to see the reflection in action. The original function is on the right, and $y=-\sqrt{x}$ is on the right. A parabola has a null horizontal reflection, but it can do a vertical reflection: $y=-x^2$. A curious thing is that graphs that are origin symmetric, like the graph of $y=\frac{1}{x}$. This is to be expected, because if we simplify the equation with the horizontal reflection $y=\frac{1}{(-x)}$, we get $y=-\frac{1}{x}$, which is the same as the vertical reflection. Go back to the last page and you can see the vertical reflection on the reciprocal graph.









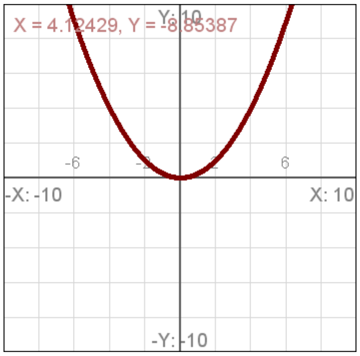


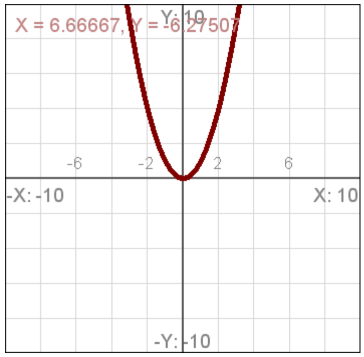
For a graph like the square root graph, you can apply both transformations simultaneously, and get a completely unique graph. $y=-\sqrt{-x}$ does not simplify as the negative cannot pass through the radical, producing the graph shown here.

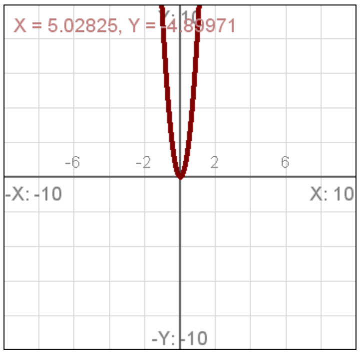
To describe these tranformations in function notation, the horizontal reflection is $f(-x)$, and the vertical reflection is $-f(x)$.

To apply these transformation to a series of points, apply the horizontal reflection by changing the sign on the *x*-coordinate, apply the vertical reflection by changing the sign on the *y*-coordinate.

Reflections are achieved by essentially multiplying by a negative one. If be multiply by some other value we get **stretching or compressing** effects, depending whether the multiplier is on the interval (0,1) or on (1,∞).



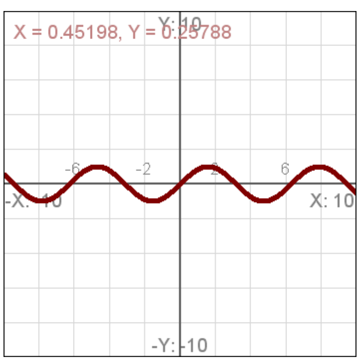


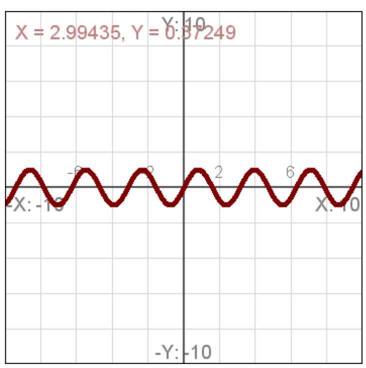


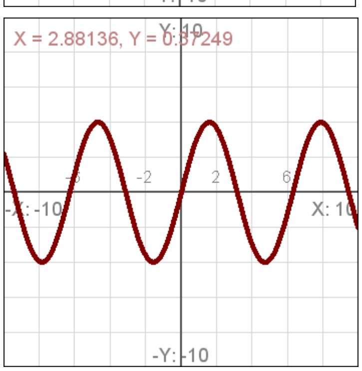
Horizontal stretching and compressing, like with horizontal shift, is applied "inside" the function, replacing *x* with *ax*, where *a* is the multiplier. If we wish to compress the graph in the horizontal direction, *a* should be on the interval (1,∞), i.e. bigger than 1. If we wish to stretch the graph horizontally, *a* should be on the interval (0,1), i.e. less than 1. For example: to stretch the parabola horizontally by a factor of two, we are going to multiply by $\frac{1}{2}$, inside any operations of the function: $y=\left(\frac{1}{2}x\right)^2$. To compress the graph by a factor of 3, replace *x* with 3*x*: $y=(3x)^2$.

Vertical stretching and compressing works more the way you would expect. If you wish to stretch the graph vertically (make it bigger in *y*), you would multiply by a number bigger than 1. If you wanted to vertically compress the graph (make it smaller in *y*) you would multiply by a number less than 1. Notice that for functions like the parabola, if we simplify the first equation we get $y=\frac{1}{4}x^2$, so that stretching the graph here vertically by 2, is the same as compressing it vertically by $\frac{1}{4}$. Similarly with the horizontal compression, it simplifies to $y=9x^2$, so this is the same as a vertical stretch by a factor of 9. And we can see that in the graphs.

In many of our functions, as was stated at the top, vertical stretching/compressing have a similar effect to horizontal stretching/compressing. From looking at the graph, they can't really be differentiated. However, we can see with the graph of $y=\sin(x)$, that they can have different effects with the right function. Compare the original graph with $y=\sin(2x)$ (top right) and $y=4\sin(x)$ (bottom left). Notice that the sine graph has a number of zeros that repeat over and over. The horizontal compression (2) makes the zeros appear closer together (it reduces the interval by a factor of 2), but it does not stretch the graph vertically at all (the maximum/minimum is still ±1). Whereas the vertical stretch (4) increases the height of the maximum/minimum (to ±4), but does not move the zeros either left or right. You will study the properties of this function when you do the trigonometry section of precalculus.





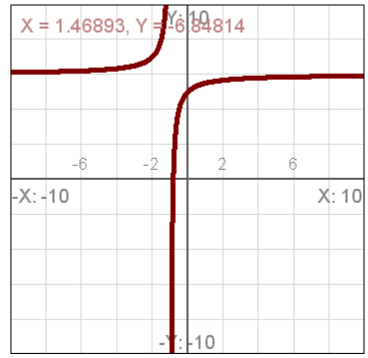


To express a horizontal stretch or compression in function notation, we write $f(ax)$, and a vertical stretch as $af(x)$.

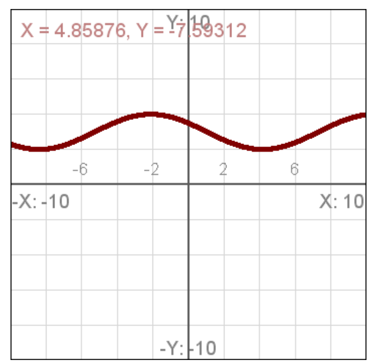
To apply a stretch/compression the a series of points, apply the horizontal stretch/compress to the *x*-coordinate only; to apply a vertical stretch/compression to the *y*-coordinate only.

Examples.

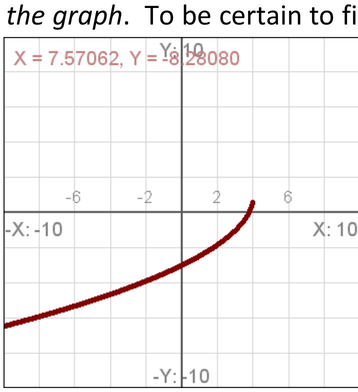
a. *Apply the following tranformations to the graph of* $f(x)=\frac{1}{x}$*: i) shift left by 1, ii) reflect vertically, iii) shift up by 6. Sketch the graph*. In function notation, we can write this as $-f(x+1)+6$, and so our transformed graph is $g(x)=-\frac{1}{x+1}+6$.



b. *Apply the following transformations to the graph of* $f(x)=\sin(x)$*: i) horizontal stretch by 2, ii) horizontal shift right by 1, iii) vertical reflection, iv) vertical shift by 3. Sketch the graph*. We will let our calculator sketch the graph, but we need to come up with the equation. Remember that all horizontal changes are "opposite", so our horizontal stretch is a $\frac{1}{2}$ multiplier inside the function, and a horizontal shift right of 1 means subtract 1. The horizontal shifts alone give us $f\left(\frac{1}{2}(x-1)\right)$, and then applying the vertical changes outside the function, we apply first the reflection, and then the shift: $-f\left(\frac{1}{2}(x-1)\right)+3$. Applying this to the sine function gives us $g(x)=-\sin\left(\frac{1}{2}(x-1)\right)+3$.



c. *List the transformations applied to the function* $f(x)=-2\sqrt{-x+4}+1$. *Compare your list to the graph*. To be certain to find the correct transformations, we have to write the function with the transformations isolated, particularly under the square root. Here, we want to factor out any coefficients of *x*: $f(x)=-2\sqrt{-(x-4)}+1$. We have i) a vertical shift up of 1 (+1 outside the root), ii) a vertical stretch of 2 (2 outside the root), iii) a vertical reflection (negative outside the root), iv) a horizontal reflection (negative under the root), and v) a horizontal shift of 4 to the right (-4 under the root).



d. *Consider the points (3,2), (1,1), and (-1, 4). Apply the following transformations to these points: i) horizontal reflection, ii) horizontal shift left by 2, iii) vertical stretch by 4, and vertical shift down 3*. Under horizontal reflection, our three points become (-3,2), (-1,1), and (1,4). The horizontal shift left by 2 makes them (-5,2), (-3,1), and (-1,4). Notice that both threse transformations only changed the *x*-coordinate. Now we tackle the *y*-coordinate. A vertical stretch by 4 means multiply by 4, giving us (-5,8), (-3,4), and (-1,16). And the vertical shift down by 3 gives us, finally, (-5,5), (-3,1), and (-1,13).

Problems.

i. Shift the graph $f(x) = x^3$ left by 4. Sketch the original graph and the transformed graph.

ii. Shift the graph $f(x) = \frac{1}{x}$ down by 2, and reflect horizontally. Sketch the original graph and the transformed graph. Give the new equation.

iii. Shift the graph $f(x) = x$ right by 3, stretch vertically by 5, and reflect vertically. Sketch the original graph and the transformed graph. Give the new equation.

iv. Shift the graph $f(x) = \sqrt{x}$ vertically up by 6, horizontally left by 4, reflect horizontally, reflect vertically, and stretch horizontally by a factor of 3. Sketch the original graph and the transformed graph. Give the new equation.

v. Shift the points (0,0), (1,3), (2,5), (4,12) horizontally right by 1, reflect vertically, compress vertically by 2.

vi. List the transformations applied to the function $4f(-x+3)$.

vii. List the transformations applied to the function $-f(x+1)-4$.

viii. List the transformation applied to the function $-\frac{1}{2}f\left(3(x-1)\right)+5$.

ix. List the transformations applied to the equation $2(x-3)^2 + 4(y+5) = 8$ [Hint: remember that this isn't a function! Both variables behave the same.]

x. How do the tranformations on (iv) change the domain and range of the function?