9/5/2024

Linear and Quadratic Functions and Applications

Linear functions as transformations

$$y = x$$

Identity function is the base linear function

$$y = x + 3$$

Think of this as a horizontal shift left by 3, or a vertical shift up by 3.

Similarly

$$y = 2x$$

Think of this as a horizontal compression (by 2) or a vertical stretch (by 2)

Normally, we think about linear functions in terms of their slope and their intercept

$$y = mx + b$$

m is the slope and b is the y-intercept (what the value of y is when x=0… where the line crosses the y-axis).

You need only two points to plot a line. The slope-intercept form typically you would plot the y-intercept at (0,b), and then use the slope (over one unit in x, up (or down) in y the value of m) to obtain a second point, and then connect the dots.

We can calculate the value of the slope if we have two points:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} = \frac{rise}{run}$$

The slope is a rate of change: how much does y change for each unit of x.

It is possible to find the equation of the line using the slope-intercept form…

Find the equation of the line connecting the points (2,4) and (5,9)

$$m = \frac{9 - 4}{5 - 2} = \frac{5}{3}$$

$$4 = \left(\frac{5}{3}\right)(2) + b$$

$$4 = \frac{10}{3} + b$$

$$b = 4 - \frac{10}{3} = \frac{12}{3} - \frac{10}{3} = \frac{2}{3}$$

$$y = \left(\frac{5}{3}\right)x + \frac{2}{3}$$

Point-slope form:

$$y - y_0 = m(x - x_0)$$

$$y - 9 = \frac{5}{3}(x - 5)$$

$$y - 9 = \frac{5}{3}x - \frac{25}{3}$$

$$y = \frac{5}{3}x - \frac{25}{3} + 9 = \frac{5}{3}x - \frac{25}{3} + \frac{27}{3} = \frac{5}{3}x + \frac{2}{3}$$

$$y = \frac{5}{3}x + \frac{2}{3}$$

Standard form:

$$Ax + By = C$$

This form can be useful when doing plots of lines because the intercepts are easy to find.

$$2x + 3y = 12$$

X-intercept occurs when y=0

$$2x = 12$$
$$x = 6$$

$$(6,0)$$

Y-intercept occurs when x=0

$$3y = 12$$
$$y = 4$$

$$(0,4)$$

Horizonal line is still a function, y=b (slope is 0)

$$f(x) = b, y = b$$

The vertical line is not a function, x=a
A vertical line fails the vertical line test. The slope of a vertical line is undefined: division by 0

Related to the idea of slope is the average rate of change

The slope of the a line that connects two points on function (line=secant line)

$$\big(a, f(a)\big), \big(b, f(b)\big)$$

$$\frac{f(b) - f(a)}{b - a}$$

$$f(x) = x^2 + 1$$

Find the average rate of change between x=2, x=4

$$f(2) = 2^2 + 1 = 5$$
$$f(4) = 4^2 + 1 = 17$$

$$(2,5), (4,17)$$

$$\frac{17 - 5}{4 - 2} = \frac{12}{2} = 6$$

On average, for each unit increase in x, the y-value increases by 6.

Absolute values

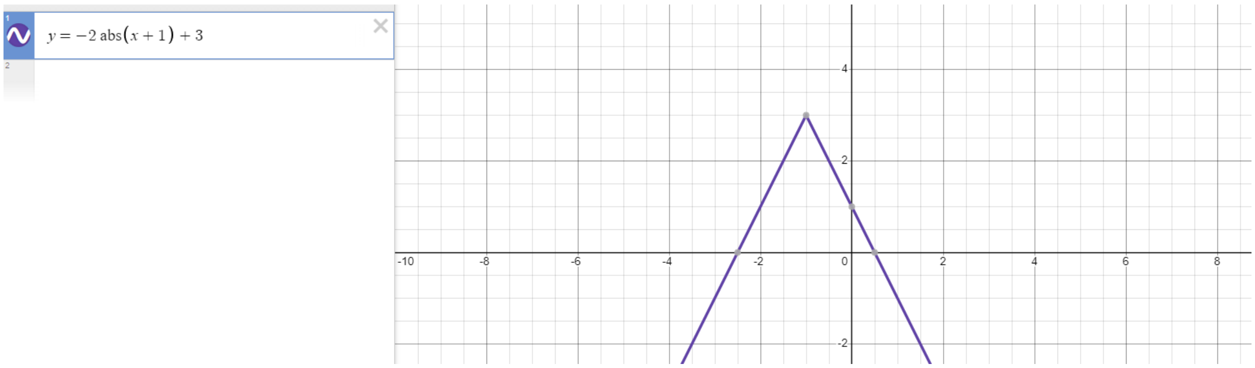
$$|a| = \begin{cases} a, a \geq 0 \\ -a, a < 0 \end{cases}$$

$$|3| = 3$$
$$|-7| = 7 = -(-7)$$

$$f(x) = |x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

$$f(x) = -2|x+1| + 3$$

Here I've applied a horizontal shift left of 1, a vertical stretch of 2, a vertical reflection, and a vertical shift up of 3



Quadratic Functions

Two main forms:

$$f(x) = ax^2 + bx + c$$
$$f(x) = a(x-h)^2 + k$$

(h, k) is the vertex.

Converting from the quadratic (standard) form to the vertex form requires completing the square.

$$h = -\frac{b}{2a}, k = f(h)$$

To plot from the vertex form, plot the vertex and one other point (the graph will be symmetric around the line x=h, the axis of symmetry)

To plot the regular quadratic function, to find the y-intercept (set x=0)… the point (0,c)
Then also find the x-intercepts. Here, set y=0

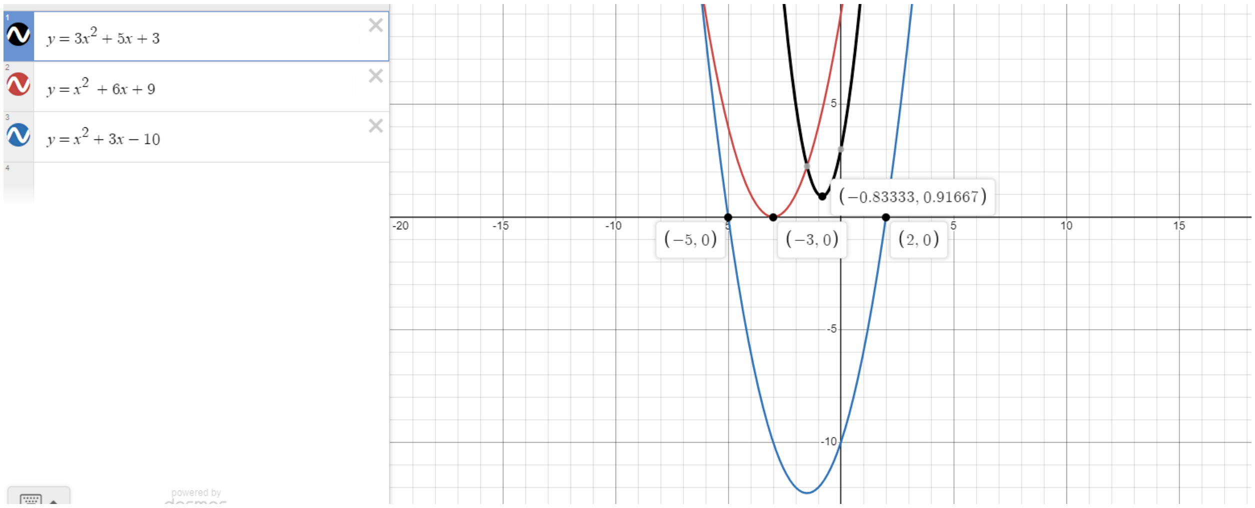
$$0 = ax^2 + bx + c$$

Then factor or use the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If the number under the square root is negative, there are no real solutions (therefore, not x-intercepts).
If that happens, just guess some additional x-values, plug into the equation and plot from there.

If the discriminant ($b^2 - 4ac$) is negative, there are no real solutions (two complex solution)
If the discriminant is positive, there are two real solutions
If the discriminant is exactly 0, then the quadratic is a perfect square and there is only one real solution



Increasing and decreasing

Increasing: positive rate of change, decreasing: negative rate of change
Change in direction happens at the turning point (vertex)

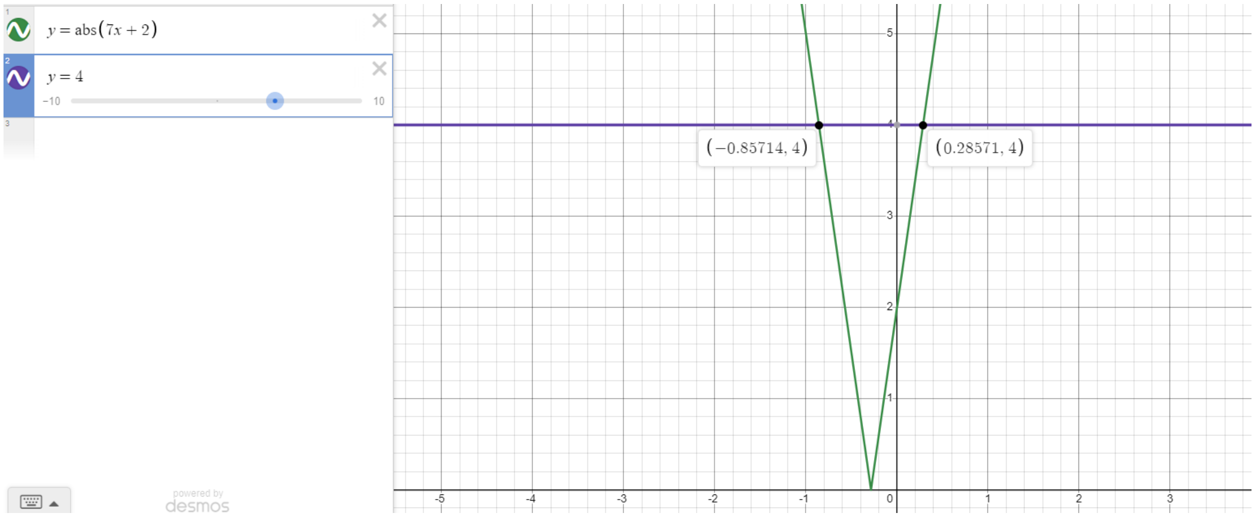
On what interval is the graph increasing: (vertex x-coordinate, infinity)
Decreasing: (-infinity, vertex x-coordinate)

Vertex itself, the y-coordinate will represent the minimum for an upward opening parabola, and the maximum for a downward opening parabola.

Inequalities with functions:

$$f(x) < g(x)$$

$$|7x + 2| < 4$$



$y = \text{abs}(7x + 2)$
$y = 4$
−10
10
(−0.85714, 4)
(0.28571, 4)
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