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Division of Polynomials: Long Division, Synthetic Division
Remainder and Factor Theorems
Rational Zeros, Decartes' Rule of Signs, Real Zeros

Long division of polynomials
Is an extension of long division of numbers

$$\frac{1469}{17}$$

$$\begin{array}{r|l} & 86 \text{ R } 7 \\ 17 & \boxed{146}9 \\ & -136 \\ \hline & 169 \\ & -102 \\ \hline & 7 \end{array}$$

$$\begin{array}{r} {}^{5}17 \\ \times\ 8 \\ \hline 136 \end{array} \qquad \begin{array}{r} {}^{3}17 \\ \times\ 5 \\ \hline 85 \\ 17 \\ \hline 102 \end{array}$$

The algorithm is similar: the leading term of the large polynomial and divide by the leading terms of the dividing polynomial. Multiply and subtract, bring down the next term and continue until the remainder is a smaller degree than the divider.

$$\frac{x^3 - 4x^2 + 2x - 6}{x - 2}$$

$$
\begin{array}{r|l}
 & x^2 - 2x - 2 \\
\hline
x - 2 & x^3 - 4x^2 + 2x - 6 \\
 & -x^3 + 2x^2 \\
\hline
 & \quad -2x^2 + 2x \\
 & \quad -(+2x^2 \mp 4x) \\
\hline
 & \qquad -2x - 6 \\
 & \qquad +2x \mp 4 \\
\hline
 & \qquad\qquad -10
\end{array}
$$

$\frac{x^3}{x} =$

$\frac{-2x^2}{x}$

$\frac{-2x}{x}$

$$x^3 - 4x^2 + 2x - 6 = (x - 2)(x^2 - 2x - 2) + (-10)$$

$$\frac{x^3 - 4x^2 + 2x - 6}{x - 2} = x^2 - 2x - 2 + \frac{-10}{x - 2}$$

$$x^3 - 4x^2 + 2x - 6 = dividend\ (the\ thing\ being\ divided\ into)$$
$$(x - 2) = divisor\ (the\ thing\ doing\ the\ division)$$
$$x^2 - 2x - 2 = quotient\ (the\ \text{"whole"}\ part\ of\ the\ dividing\ process)$$
$$-10 = remainder\ (the\ left\ over\ when\ division\ can't\ continue)$$

Example.

$$\frac{x^5 - 243}{x - 3}$$

$$x^4 + 3x^3 + 9x^2 + 27x + 81$$

$$x-3 \overline{) \, x^5 + 0x^4 + 0x^3 + 0x^2 + 0x - 243}$$

$$-x^5 + 3x^4$$

$$3x^4 + 0x^3$$

$$-3x^4 + 9x^3$$

$$9x^3 + 0x^2$$

$$-9x^3 + 27x^2$$

$$27x^2 + 0x$$

$$-27x^2 + 81x$$

$$81x - 243$$

$$-81x + 243$$

$$0$$

$$\frac{6x^4 - 5x^2 + 12}{2x^2 - x + 1}$$

$$
\begin{array}{r}
3x^2 + \frac{3}{2}x - \frac{13}{4} \\
2x^2 - x + 1 \overline{)\, 6x^4 + 0x^3 - 5x^2 + 0x + 12} \\
-6x^4 + 3x^3 - 3x^2 \\
\hline
3x^3 - 8x^2 + 0x \\
-3x^3 + \frac{3}{2}x^2 - \frac{3}{2}x \\
\hline
-\frac{13}{2}x^2 - \frac{3}{2}x + 12 \\
+\frac{13}{2}x^2 - \frac{13}{4}x + \frac{13}{4} \\
\hline
-19/4\,x + 61/4
\end{array}
$$

$$\frac{6x^4}{2x^2} \qquad \frac{3x^3}{2x^2} \qquad \frac{-\frac{13}{2}x^2}{2x^2}$$

$$3x^2 + \frac{3}{2}x - \frac{13}{4} + \frac{-\frac{19}{4}x + \frac{61}{4}}{2x^2 - x + 1}$$

Synthetic Division
Dividing a polynomial by a “factor” of the form $(x - c)$

General form is:

$$
\begin{array}{c|l}
c & \text{coeffients} \\
 & c\cdot\# \\
\hline
 & \# \quad \text{add}
\end{array}
$$

Example:

$$\frac{x^3 - 4x^2 + 2x - 6}{x - 2}$$

$$
\begin{array}{r|rrrr}
2 & 1 & -4 & 2 & -6 \\
  &   & 2 & -4 & -4 \\
\hline
  & 1 & -2 & -2 & \boxed{-10}
\end{array}
\quad \text{Remainder}
$$

$$x^2-2x-2 \quad + \frac{-10}{x-2}$$

Example.

$$\frac{x^5-243}{x-3}$$

$$
\begin{array}{r|rrrrrr}
3 & 1 & 0 & 0 & 0 & 0 & -243 \\
  &   & 3 & 9 & 27 & 81 & 243 \\
\hline
  & 1 & 3 & 9 & 27 & 81 & \boxed{0}
\end{array}
\quad R
$$

$$x^4+3x^3+9x^2+27x+81$$

Since you have to divide with $(x-c)$ form, if you have a factor like $(2x-3)$, you need to pull out the coefficient of x and divide with $2\left(x-\frac{3}{2}\right)$. Divide out the 2 separately after the fact.

Remainder and Factor Theorems

Remainder Theorem: when you divide a polynomial by $(x-c)$, the value of the remainder is the same as the value of $f(c)$.

$$\frac{x^3-4x^2+2x-6}{x-2}$$

The remainder I got was -10.
What is the value of $p(2)$ if $p(x) = x^3 - 4x^2 + 2x - 6$?

$$p(2) = 2^3 - 4(2)^2 + 2(2) - 6 = 8 - 16 + 4 - 6 = -8 + 4 - 6 = -4 - 6 = -10$$

What if the remainder is zero?
The Factor Theorem: Then the divisor is a factor of the polynomial, and c is a zero of the polynomial.

Rational Zeros Theorem:
Reduces our guesses for possible rational zeros (therefore factors) of a polynomial based on the leading coefficient and the final constant.

$$3x - 2$$

$$3x - 2 = 0$$
$$3x = 2$$
$$x = \frac{2}{3}$$

factor of constant

factor of leading coeff.

$$ax^n + bx^{n-1} + \cdots yx + z$$

In this example, a is the leading coefficient, and z is the final constant

p is the set of factors of a
q is the set of factors of z

Possible rational zeros:

$$\pm\frac{q}{p}$$

$$6x^4 + 7x^3 - 11x^2 + 2x - 35$$

$$p = 1,2,3,6$$
$$q = 1,5,7,35$$

$$\pm\frac{q}{p} = \pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}, \pm\frac{5}{1}, \pm\frac{5}{2}, \pm\frac{5}{3}, \pm\frac{5}{6}, \pm\frac{7}{1}, \pm\frac{7}{2}, \pm\frac{7}{3}, \pm\frac{7}{6}, \pm\frac{35}{1}, \pm\frac{35}{2}, \pm\frac{35}{3}, \pm\frac{35}{6}$$

Typically, if you have to test these by hand, start close to 0 and work your way out, you may also want to start with integers before doing fractions

Descartes' Rule of Signs

For positive real zeros: the number of sign changes in a polynomial indicates the maximum number of possible positive zeros (or if less, but units of 2… if the max is 5, then it could 3 or 1, not 4 or 2).

For negative real zeros, replace x with -x, and then count the resulting sign changes (the maximum number of negative zeros is the number of sign changes in the polynomial).

$$6x^4 + 7x^3 - 11x^2 + 2x - 35$$

For example.
3 sign changes, or there are either 3 positive zeros or 1 positive zero.

For the negative zeros: replace x with -x

$$6x^4 - 7x^3 - 11x^2 - 2x - 35$$

There is only one sign change, so there is only one negative real zero.

Fundamental Theorem of Algebra:
Any polynomial of degree n, can have no more than n zeros. And specifically, if we count both real and complex zeros (including multiplicities), a polynomial of degree n has exactly n zeros.