9/12/2024

One-to-One Functions
Inverses

Relation is just a way to represent a relationship between variables, between inputs and outputs. Functions are relations where each input has only one corresponding output. The visual test for this was the vertical line test. Any vertical line (which represents a single value of x (input)) can only cross the graph one time.

Every relation has an inverse. Swap the domain and range, and then switch the direction of the arrow.

If a relation is defined by pairs of points (x,y), the inverse relation is defined by pairs of points (y,x).

When we start talking about functions, we don't just want our inverses to be relations, we want them to be functions.

Our goal will be to determine if a given function has an inverse **function**, and if it exists, find an expression for it. (If it does not exist, is there a way to restrict the domain of the original function, so that its inverse will also be a function.)

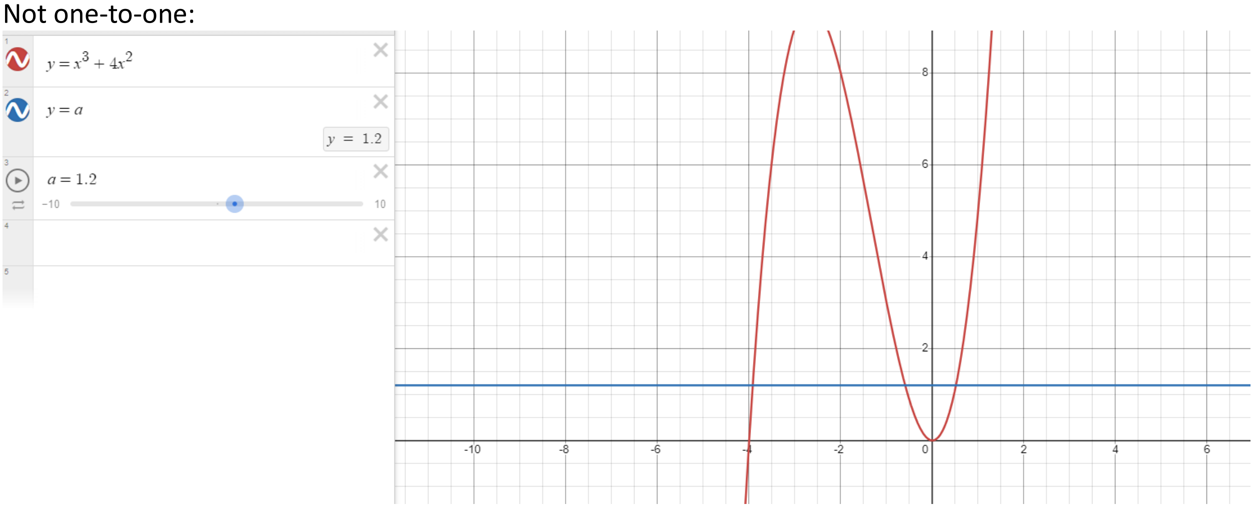
A one-to-one function is a function where for each input there is just a single output AND for every output, there is only one input that gets you there. Each output corresponds to a unique input. Each x maps to only one y, and each y maps to only one x.

Graphically, the visual test is called the horizontal line test: if the graph crosses every horizontal line only once, then the inverse is a function.

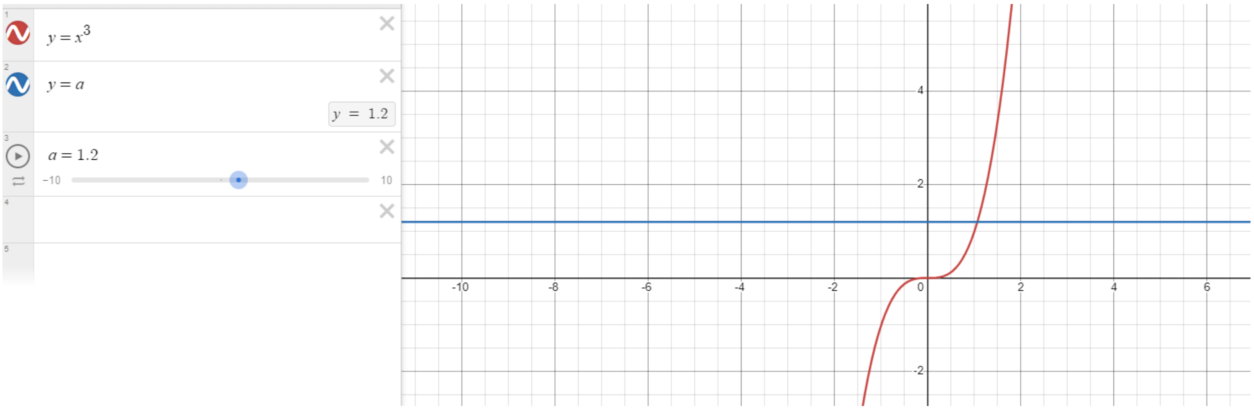
In a list, if there are any repeated y-values, then the function is not one-to-one.

Typically, expect that odd functions are one-to-one, or no-symmetry functions. But even functions are not one-to-one without a domain restriction.

Not one-to-one:



One-to-one:



Notation for inverse functions:
Inverse function for $f(x)$ is given by $f^{-1}(x)$

This -1 in the position of an exponent is not the same as the negative one exponent. This is not equivalent to $\frac{1}{f(x)} \neq f^{-1}(x)$

If you compose the original function with the inverse function (in either order), the result will be the input (x).

$$f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$$

An inverse is a reverse operation. Whatever f did to the input, f^{-1} undoes it.

$$y = x^3, f(x) = x^3$$

The inverse of the cube function is the cube root function.

$$y = \sqrt[3]{x}, f^{-1}(x) = \sqrt[3]{x}$$

Think about how we define the cube root: we define it in terms of the cube. What number, if I cube it, will get me the number under the cube root?

We can define the square root that way as well, as long as we are only dealing with positive numbers.

$$\left(\sqrt[3]{x}\right)^3 = \sqrt[3]{x^3} = x$$

To find the inverse algebraically for an equation that is one-to-one, follow these steps:

1) Replace x in the equation with y, and replace y with x
2) Solve for y in the new equation

Example.
Find the inverse function for $f(x) = 3x - 5$

$$y = 3x - 5$$
$$x = 3y - 5$$

$$x + 5 = 3y$$
$$y = \frac{x+5}{3}$$

$$f^{-1}(x) = \frac{x+5}{3}$$

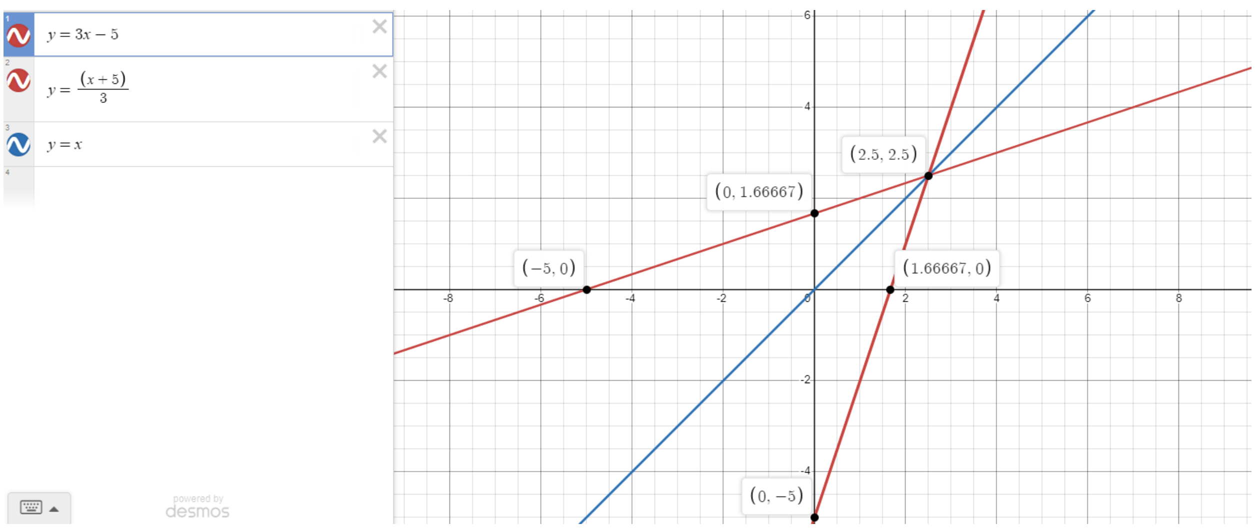
$$f\left(f^{-1}(x)\right) = 3\left(\frac{x+5}{3}\right) - 5 = x + 5 - 5 = x$$

$$f^{-1}\left(f(x)\right) = \frac{(3x-5)+5}{3} = \frac{3x}{3} = x$$

Verify that/Determine if $g(x)$ is the inverse of $f(x)$, use composition to test to see if $f\left(g(x)\right)$ reduces to x.

Another way to test if a function is the inverse of another function: since we switched the roles of x and y between the inverse and the original function, the graphs will display symmetry across the line y=x.

Create a graph of f(x), $f^{-1}(x)$, y=x.



Example.

$$f(x) = 3 - \sqrt[3]{x-2}$$

Find the inverse and verify using a graph.

$$y = 3 - \sqrt[3]{x-2}$$

$$x = 3 - \sqrt[3]{y-2}$$

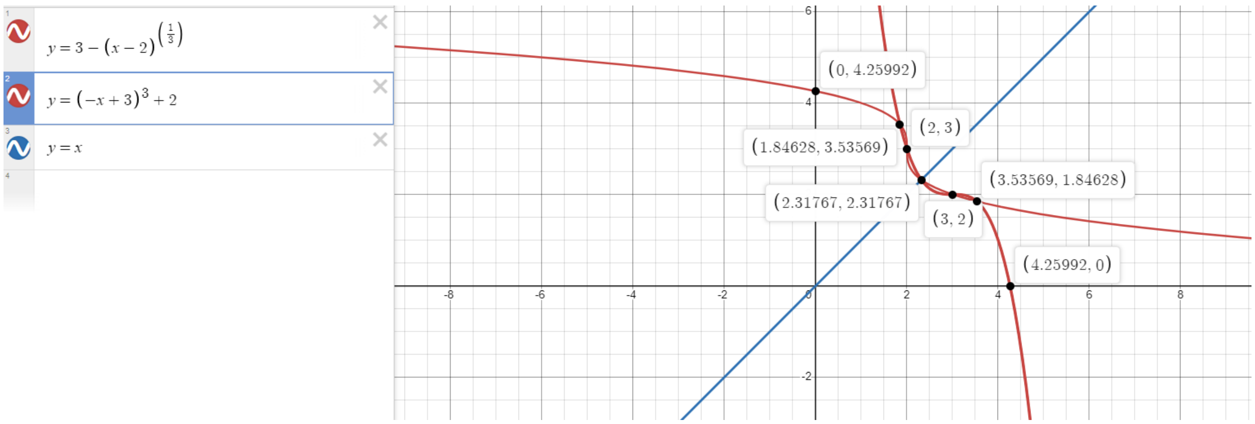
$$x - 3 = -\sqrt[3]{y-2}$$

$$3-x=\sqrt[3]{y-2}$$

$$(3-x)^3=y-2$$

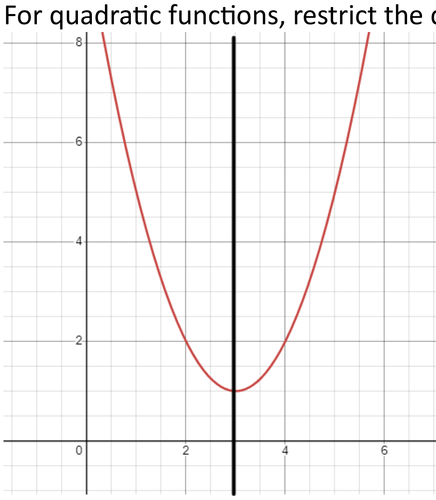
$$y=(3-x)^3+2$$

$$f^{-1}(x)=(-x+3)^3+2$$



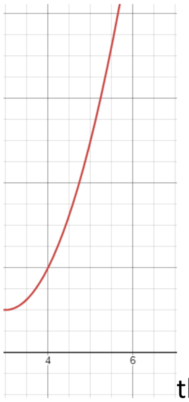
What if the function is not one-to-one? How do we know where to restrict the domain to make the inverse a function?

For quadratic functions, restrict the domain to one side of the vertex. (similarly in absolute values)



axis of symmetry of $f(x)=x^2-6x+10$ is at x=3

Restrict domain to $x\geq 3$

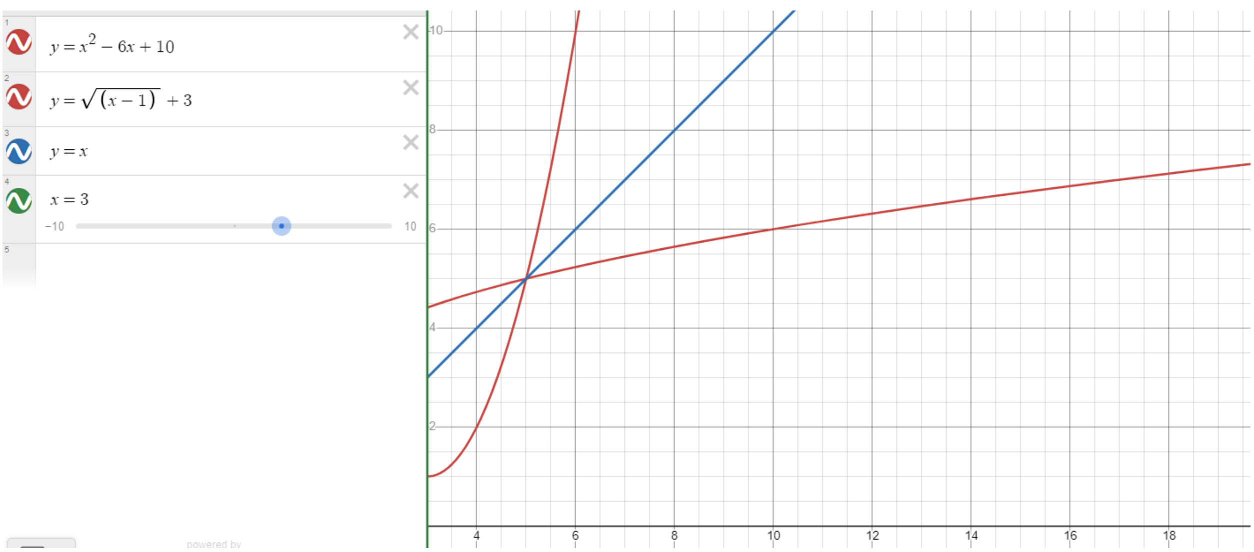


this portion of the graph is one-to-one.

$$f(x) = x^2 - 6x + 10$$
$$f(x) = (x-3)^2 + 1$$

$$y = (x-3)^2 + 1$$
$$x = (y-3)^2 + 1$$

$$x - 1 = (y-3)^2$$
$$\sqrt{x-1} = y - 3$$
$$y = \sqrt{x-1} + 3$$



Since inverses and their original functions swap the roles of x and y, there is a relationship between the domain and the range of a function and its inverse:

The domain of the original function is the range of the inverse function.
The range of the original function is the domain of the inverse function.

One way of finding the range of a function is to find its inverse, and then find the domain of that function.

$$x = \frac{y+1}{y+3}$$

$$x(y+3) = y+1$$

$$xy + 3x = y+1$$

Put all your y’s on one side of the equation and factor out the y

$$xy - y = -3x + 1$$

$$y(x-1) = -3x + 1$$

$$y = \frac{-3x+1}{x-1}$$

Review for exam #1